

## Miscellany

### 1.1 Jittering

A. Parameters of  $m$  when  $\alpha \leq 0$ .

Assume (and check later) that  $\max\left(\kappa_0, \frac{\kappa_2}{3\mu_2^2}\right) = \kappa_0$ , so that

$$J_\nu(a) = 2(1-\nu) \left(1 + \frac{1}{3\mu_2}\right) + 2\nu\kappa_0.$$

We take  $-\infty < \alpha \leq 0$ . Then

$$\begin{aligned} m(x) &= \frac{(x^2 - \alpha)}{\int_{-1}^1 (x^2 - \alpha) dx} = \frac{3(x^2 - \alpha)}{2(1 - 3\alpha)}, \\ \mu_2 &= \int_{-1}^1 x^2 m(x) dx = \frac{3 - 5\alpha}{5(1 - 3\alpha)}, \\ \kappa_0 &= \int_{-1}^1 m^2(x) dx = \frac{3(3 - 10\alpha + 15\alpha^2)}{10(1 - 3\alpha)^2}, \\ \kappa_2 &= \int_{-1}^1 x^2 m^2(x) dx = \frac{3(15 - 42\alpha + 35\alpha^2)}{70(1 - 3\alpha)^2}, \\ \frac{\kappa_2}{3\mu_2^2} &= \frac{5(15 - 42\alpha + 35\alpha^2)}{14(3 - 5\alpha)^2}, \\ J_\nu(a) &= 2 \left\{ (1-\nu) \left(1 + \frac{5(1-3\alpha)}{3(3-5\alpha)}\right) + \nu \left(\frac{3(3-10\alpha+15\alpha^2)}{10(1-3\alpha)^2}\right) \right\}. \end{aligned}$$

Then  $\alpha$  satisfies

$$0 = \frac{1}{2} J'_\nu(a) = (1-\nu) \frac{d}{da} \frac{5(1-3\alpha)}{3(3-5\alpha)} + \nu \frac{d}{da} \left( \frac{3(3-10\alpha+15\alpha^2)}{10(1-3\alpha)^2} \right),$$

so that

$$\begin{aligned} \nu &= \frac{\frac{d}{da} \frac{5(1-3\alpha)}{3(3-5\alpha)}}{\frac{d}{da} \left( \frac{5(1-3\alpha)}{3(3-5\alpha)} - \frac{3(3-10\alpha+15\alpha^2)}{10(1-3\alpha)^2} \right)} \\ &= \left( 1 + \frac{9(3-5\alpha)^2}{25(1-3\alpha)^3} \right)^{-1}. \end{aligned}$$

Thus for  $\alpha \leq 0$  we have

$$\frac{25}{106} \leq \nu = \left( 1 + \frac{9(3-5\alpha)^2}{25(1-3\alpha)^3} \right)^{-1} \leq 1;$$

the limiting cases being (i)  $\alpha = 0$ ,  $\nu = 25/106$ ,  $m(x) = 3x^2/2$  for  $-1 \leq x \leq 1$ ; and (ii)  $\alpha \rightarrow -\infty$ ,  $\nu \rightarrow 1$ ,  $m(x) \rightarrow .5$  for  $-1 \leq x \leq 1$  (the uniform density).

B. Parameters of  $m$  when  $\alpha > 0$ .

When  $\alpha > 0$ , we let  $\delta = \sqrt{\alpha}$ ,

$$\begin{aligned} m(x) &= \begin{cases} \frac{x^2 - \alpha}{b}, & \sqrt{\alpha} \leq |x| \leq 1; \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{3(x^2 - \delta^2)}{2(1-\delta)^2(1+2\delta)} = \frac{3(x^2 - \alpha)}{2(1-\sqrt{\alpha})^2(1+2\sqrt{\alpha})}, & \sqrt{\alpha} \leq |x| \leq 1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$$\text{where } b = 2 \int_{\delta}^1 (x^2 - \delta^2) dx = \frac{2}{3} (1 - \delta) (1 + \delta - 2\delta^2) = \frac{2}{3} (1 - \delta)^2 (1 + 2\delta).$$

$$\begin{aligned} \mu_2 &= \int_{-1}^1 x^2 m(x) dx = \frac{2}{b} \int_{\delta}^1 x^2 (x^2 - \delta^2) dx = \frac{2}{15b} (3 - 5\delta^2 + 2\delta^5) \\ &= \frac{2}{15b} (1 - \delta)^2 (3 + 6\delta + 4\delta^2 + 2\delta^3) = \frac{3 + 6\delta + 4\delta^2 + 2\delta^3}{5(1 + 2\delta)}, \end{aligned}$$

$$\begin{aligned} k_0 &= \int_{-1}^1 m^2(x) dx = \frac{2}{b^2} \int_{\delta}^1 (x^2 - \delta^2)^2 dx = \frac{2}{15b^2} (3 - 10\delta^2 + 15\delta^4 - 8\delta^5) \\ &= \frac{2}{15b^2} (1 - \delta)^3 (3 + 9\delta + 8\delta^2) = \frac{3(3 + 9\delta + 8\delta^2)}{10(1 - \delta)(1 + 2\delta)^2}, \end{aligned}$$

$$\begin{aligned} k_2 &= \int_{-1}^1 x^2 m^2(x) dx = \frac{2}{b^2} \int_{\delta}^1 x^2 (x^2 - \delta^2)^2 dx = \frac{2}{105b^2} (15 - 42\delta^2 + 35\delta^4 - 8\delta^7) \\ &= \frac{2}{105b^2} (1 - \delta)^3 (15 + 45\delta + 48\delta^2 + 24\delta^3 + 8\delta^4) \\ &= \frac{3}{70} \frac{15 + 45\delta + 48\delta^2 + 24\delta^3 + 8\delta^4}{(1 - \delta)(1 + 2\delta)^2}. \end{aligned}$$

$$\begin{aligned} \frac{1}{2} J_v(\delta) &= (1 - v) \left( 1 + \frac{1}{3\mu_2} \right) + vk_0 \\ &= (1 - v) \left( 1 + \frac{5(1 + 2\delta)}{3(3 + 6\delta + 4\delta^2 + 2\delta^3)} \right) + v \left( \frac{3}{10} \frac{(3 + 9\delta + 8\delta^2)}{(1 - \delta)(1 + 2\delta)^2} \right), \end{aligned}$$

$$\frac{1}{2} J'_v(\delta) = (1 - v) \left[ -\frac{10}{3} \left( \frac{\delta(4 + 7\delta + 4\delta^2)}{(3 + 6\delta + 4\delta^2 + 2\delta^3)^2} \right) \right] + v \left[ \frac{3}{10} \left( \frac{4\delta(4 + 7\delta + 4\delta^2)}{(1 - \delta)^2(1 + 2\delta)^3} \right) \right]$$

$$0 = (1 - v) \left[ \frac{-10}{3(3 + 6\delta + 4\delta^2 + 2\delta^3)^2} \right] + v \left[ \left( \frac{6}{5(1 - \delta)^2(1 + 2\delta)^3} \right) \right]$$

$$v = \left[ 1 + \frac{9}{25} \frac{(3 + 6\delta + 4\delta^2 + 2\delta^3)^2}{(1 - \delta)^2(1 + 2\delta)^3} \right]^{-1},$$

as  $\alpha \rightarrow 0$ ,  $v \rightarrow \frac{25}{106}$ ; as  $\alpha \rightarrow \pm 1$ ,  $v \rightarrow 0$ ,  $m(x) \rightarrow +\infty$ . classical optimal design

$$0 < v = \left[ 1 + \frac{9}{25} \frac{(3 + 6\sqrt{\alpha} + 4\alpha + 2\alpha\sqrt{\alpha})^2}{(1 - \sqrt{\alpha})^2(1 + 2\sqrt{\alpha})^3} \right]^{-1} < \frac{25}{106}$$

$$I_v(\xi) = (1 - v) \left( 1 + \frac{5(1 + 2\sqrt{\alpha})}{3(3 + 6\sqrt{\alpha} + 4\alpha + 2\alpha\sqrt{\alpha})} \right) + v \left( \frac{3}{10} \frac{(3 + 9\sqrt{\alpha} + 8\alpha)}{(1 - \sqrt{\alpha})(1 + 2\sqrt{\alpha})^2} \right).$$

C. Check  $\max\left(k_0, \frac{k_2}{3\mu_2^2}\right) = k_0$ .

When  $\alpha \leq 0$ ,

$$k_0 = \frac{3(3 - 10\alpha + 15\alpha^2)}{10(1 - 3\alpha)^2} = \frac{1}{2} + \frac{2}{45(\alpha - 1/3)^2},$$

$$\frac{k_2}{3\mu_2^2} = \frac{5(15 - 42\alpha + 35\alpha^2)}{14(3 - 5\alpha)^2} = \frac{1}{2} + \frac{6}{175(\alpha - 3/5)^2}, \text{ and}$$

$$k_0 - \frac{k_2}{3\mu_2^2} = \frac{6}{175} \left[ \frac{1}{(\alpha - 1/3)^2} - \frac{1}{(\alpha - 3/5)^2} \right] + \frac{16}{1575} \frac{1}{(\alpha - 1/3)^2} > 0.$$

When  $\alpha > 0$ ,

$$\Delta = \frac{k_2}{3\mu_2^2} = \frac{5}{14} \frac{15 + 45\delta + 48\delta^2 + 24\delta^3 + 8\delta^4}{(1 - \delta)(3 + 6\delta + 4\delta^2 + 2\delta^3)^2},$$

as  $\alpha \rightarrow 0$ ,  $k_0/\Delta \rightarrow \frac{189}{125}$ ; as  $\alpha \rightarrow 1$ ,  $k_0/\Delta \rightarrow 3$ .

$$\frac{k_0}{\Delta} = \frac{21}{25} \frac{(3 + 9\delta + 8\delta^2)(3 + 6\delta + 4\delta^2 + 2\delta^3)^2}{(1 + 2\delta)^2(15 + 45\delta + 48\delta^2 + 24\delta^3 + 8\delta^4)} > 1,$$

$$1.512 = \frac{189}{125} < \frac{k_0}{\Delta} < 3.$$

$$k_0 > \Delta.$$

$$\max\left(k_0, \frac{k_2}{3\mu_2^2}\right) = k_0, \text{ for all } \alpha.$$

D. Least favourable contaminant for  $\xi$ .

We have

$$\begin{aligned} \mathbf{K}_\xi &= \kappa_0 \oplus \kappa_2, \\ \mathbf{A} &= 2 \oplus 2/3, \\ \mathbf{M}_\xi &= 1 \oplus \mu_2, \\ \mathbf{H}_\xi &= \frac{1}{2} (1 \oplus 3\mu_2^2), \\ \mathbf{G}_\xi &= \left(\kappa_0 - \frac{1}{2}\right) \oplus \left(\kappa_2 - \frac{3\mu_2^2}{2}\right) \end{aligned}$$

and so

$$\begin{aligned} \mathbf{r}(\mathbf{x}) &= \frac{\tau}{\sqrt{n}} \left[ \left(\kappa_0 - \frac{1}{2}\right)^{-1/2} \oplus \left(\kappa_2 - \frac{3\mu_2^2}{2}\right)^{-1/2} \right] \begin{pmatrix} m(x) - \frac{1}{2} \\ x \left(m(x) - \frac{3\mu_2}{2}\right) \end{pmatrix} \\ \psi_\beta(x) &= \frac{\tau}{\sqrt{n}} \begin{pmatrix} \frac{m(x) - \frac{1}{2}}{\sqrt{\left(\kappa_0 - \frac{1}{2}\right)}} \\ \frac{x \left(m(x) - \frac{3\mu_2}{2}\right)}{\sqrt{\left(\kappa_2 - \frac{3\mu_2^2}{2}\right)}} \end{pmatrix}' \boldsymbol{\beta}, \end{aligned}$$

where  $\boldsymbol{\beta}$  is the unit eigenvalue corresponding to the maximum eigenvalue of  $2(\kappa_0 \oplus \kappa_2)$ . Since  $\max(\kappa_0, \kappa_2) = \kappa_0$ , we have  $\boldsymbol{\beta} = (1, 0)'$  and so

$$\psi_\beta(x) = \frac{\tau}{\sqrt{n}} \frac{m(x) - \frac{1}{2}}{\sqrt{\left(\kappa_0 - \frac{1}{2}\right)}},$$

orthogonal to  $(1, x)'$  and with  $\int_{-1}^1 \psi_{\beta}^2(x) = \tau^2/n$ . So the least favourable contaminant is  $(\tau/\sqrt{n})\psi_0(x)$  with (for  $\alpha \leq 0$ )

$$\psi_0(x) = \frac{m(x) - \frac{1}{2}}{\sqrt{(\kappa_0 - \frac{1}{2})}} = \sqrt{\frac{5}{8}}(3x^2 - 1).$$

For this, as must be the case,

$$\text{QIMSE}(\xi|\psi_0) = 2(1 - \nu) \left(1 + \frac{1}{3\mu_2(\xi)}\right) + \nu \left(1 + \frac{5}{4}(3\mu_2(\xi) - 1)^2\right) = I_\nu(\xi).$$

#### E. Maximum loss $I_\nu(\Phi)$ .

Similar to the calculation of  $I_\nu(\xi_0)$ , we have that

$$I_\nu(\Phi_n) = 2(1 - \nu) \left(1 + \frac{1}{3\lambda_2}\right) + 2\nu \max\left(\delta_0, \frac{\delta_2}{3\lambda_2^2}\right),$$

where

$$\begin{aligned} \lambda_2 &= \int_{-1}^1 x^2 \phi_n(x; c) dx \\ &= \frac{1}{2c} \sum_{i=1}^n \int_{-1}^1 x^2 I\left[t_i - \frac{c}{n} \leq x \leq t_i + \frac{c}{n}\right] dx \\ &= \frac{1}{2c} \sum_{i=1}^n \int_{t_i - \frac{c}{n}}^{t_i + \frac{c}{n}} x^2 dx \\ &= \frac{1}{6c} \sum_{i=1}^n \left[\left(t_i + \frac{c}{n}\right)^3 - \left(t_i - \frac{c}{n}\right)^3\right] \\ &= \frac{1}{n} \sum_{i=1}^n t_i^2 + \frac{c^2}{3n^2}. \end{aligned}$$

Since  $\phi_n^2(x; c) = \frac{1}{2c}\phi_n(x; c)$ , we have

$$\begin{aligned} \delta_0 &= \int_{-1}^1 \phi_n^2(x; c) dx = \frac{1}{2c}, \\ \delta_2 &= \int_{-1}^1 x^2 \phi_n^2(x; c) dx = \frac{\lambda_2}{2c}. \end{aligned}$$

Thus

$$I_\nu(\Phi_n) = 2(1 - \nu) \left(1 + \frac{1}{3\lambda_2}\right) + \frac{\nu}{c} \max\left(1, \frac{1}{3\lambda_2}\right).$$

## 1.2 2-d clustering

### A. Derivation of $I_\nu(\Phi)$

Here  $\chi = [-2, 2] \times [-2, 2]$  and  $f(x) = (1, x_1, x_2, x_1x_2, x_1^2, x_2^2)$ . Then

$$\begin{aligned}
(i) A &= \int_{-2}^2 \int_{-2}^2 \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{pmatrix} dx_1 dx_2 \\
&= \int_{-2}^2 \int_{-2}^2 \begin{pmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \\ * & x_1^2 & x_1x_2 & x_1^2x_2 & x_1^3 & x_1x_2^2 \\ * & * & x_2^2 & x_1x_2^2 & x_1^2x_2 & x_2^3 \\ * & * & * & x_1^2x_2^2 & x_1^3x_2 & x_1x_2^3 \\ * & * & * & * & x_1^4 & x_1^2x_2^2 \\ * & * & * & * & * & x_2^4 \end{pmatrix} dx_1 dx_2 \\
&= \begin{pmatrix} 16 & 0 & 0 & 0 & 64/3 & 64/3 \\ * & 64/3 & 0 & 0 & 0 & 0 \\ * & * & 64/3 & 0 & 0 & 0 \\ * & * & * & 256/9 & 0 & 0 \\ * & * & * & * & 256/5 & 256/9 \\ * & * & * & * & * & 256/5 \end{pmatrix} \\
&= \frac{64}{45} \cdot \begin{pmatrix} 180 & 0 & 0 & 0 & 15 & 15 \\ 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 15 & 0 & 0 & 0 & 36 & 20 \\ 15 & 0 & 0 & 0 & 20 & 36 \end{pmatrix} \\
(ii) M_\Phi &= E_\Phi \begin{pmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \\ * & x_1^2 & x_1x_2 & x_1^2x_2 & x_1^3 & x_1x_2^2 \\ * & * & x_2^2 & x_1x_2^2 & x_1^2x_2 & x_2^3 \\ * & * & * & x_1^2x_2^2 & x_1^3x_2 & x_1x_2^3 \\ * & * & * & * & x_1^4 & x_1^2x_2^2 \\ * & * & * & * & * & x_2^4 \end{pmatrix} \\
&= E_\Phi \begin{pmatrix} 1 & 0 & 0 & 0 & x_1^2 & x_2^2 \\ 0 & x_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2x_2^2 & 0 & 0 \\ x_1^2 & 0 & 0 & 0 & x_1^4 & x_1^2x_2^2 \\ x_2^2 & 0 & 0 & 0 & x_1^2x_2^2 & x_2^4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda_2 & \lambda_2 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{22} & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & \lambda_4 & \lambda_{22} \\ \lambda_2 & 0 & 0 & 0 & \lambda_{22} & \lambda_4 \end{pmatrix}, \text{ for} \\
\lambda_2 &= E_\Phi [X_1^2] = E_\Phi [X_2^2], \lambda_{22} = E_\Phi [X_1^2X_2^2], \lambda_4 = E_\Phi [X_1^4] = E_\Phi [X_2^4].
\end{aligned}$$

$$(iii) K_{\Phi} = \begin{pmatrix} \kappa_0 & 0 & 0 & 0 & \kappa_2 & \kappa_2 \\ 0 & \kappa_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_{22} & 0 & 0 \\ \kappa_2 & 0 & 0 & 0 & \kappa_4 & \kappa_{22} \\ \kappa_2 & 0 & 0 & 0 & \kappa_{22} & \kappa_4 \end{pmatrix}, \text{ for}$$

$$\kappa_0 = \int_{\mathcal{X}} \phi^2(x; \nu) dx_1 dx_2, \quad \kappa_2 = \int_{\mathcal{X}} x_1^2 \phi^2(x; \nu) dx_1 dx_2 = \int_{\mathcal{X}} x_2^2 \phi^2(x; \nu) dx_1 dx_2,$$

$$\kappa_{22} = \int_{\mathcal{X}} x_1^2 x_2^2 \phi^2(x; \nu) dx_1 dx_2, \quad \kappa_4 = \int_{\mathcal{X}} x_1^4 \phi^2(x; \nu) dx_1 dx_2 = \int_{\mathcal{X}} x_2^4 \phi^2(x; \nu) dx_1 dx_2.$$