Vertical D4-D2-D0 bound states on K3 fibrations and modularity

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Abstract

An explicit formula is derived for the generating function of vertical D4-D2-D0 bound states on smooth K3 fibered Calabi-Yau threefolds. It is also shown that this formula satisfies strong modularity properties, as predicted by string theory. This leads to a new construction of vector valued modular forms which exhibits some of the features of a generalized Hecke transform.

Contents

1   Introduction

The main goal of this paper is to derive an explicit conjectural formula for all BPS degeneracies of vertical D4-D2-D0 bound states on K3 fibered Calabi-Yau threefolds X, subject to certain genericity conditions. Counting D4-D2-D0 BPS states is a natural problem in the context of IIA and M-theory compactifications, as well as IIA/heterotic duality. This problem is closely related to black hole entropy [13, 58, 59, 37, 48, 12, 14], as well as BPS algebras [26, 27]. Further ramifications include connections with quantum two dimensional Yang-Mills theory and topological strings [60, 2, 1, 28], as well as wallcrossing phenomena [45, 47, 46]. As pointed out in [59, 44, 32] and studied in more depth in [15], a different connection with topological strings is provided by the Fourier-Mukai transform on elliptic fibrations. This problem is also firmly rooted in gauge theory, since it can be regarded as a string theoretic generalization of Vafa-Witten theory [61] on a K3 surface. Exact results for this case have been obtained in [44, 36]. At the same time, this problem has received a fair amount of attention in the mathematical literature [20, 21, 57, 22, 55, 56], in the framework of Dolandson-Thomas invariants of stable sheaves on Calabi-Yau threefolds

As shown in [62, 14, 18], on physics grounds the generating function for D4-D2-D0 BPS indices is expected to have strong modularity properties. This has been confirmed by many explicit computations for stable torsion-free sheaves supported on rational surfaces [23, 64, 65, 66, 67, 24, 25, 39, 38, 40, 19, 55, 56]. These results can be placed in the context of Donaldson-Thomas theory if such sheaves are viewed as two dimensional torsion modules supported on rigid divisors in Calabi-Yau threefolds. For two dimensional sheaves supported on the fibers of a K3 pencil, modularity has been shown in certain cases in [20], as discussed in more detail below. Finally note that modularity results have been also proven in the context of curve counting theories in [10, 41, 33, 42, 51].

1.1 Partition functions

The geometric framework for this paper is that of lattice polarized K3 fibered projective Calabi-Yau threefolds satisfying certain natural genericity assumptions. In particular all fibers are assumed to be irreducible algebraic K3 surfaces with at most simple nodal singularities. In addition, one also assumes the existence of a lattice polarization defined as in [41, 33] in terms of an auxiliary smooth K3 fibration obtained by extremal transitions. Very briefly, any K3 fibration is assigned a sublattice Λ of the middle cohomology lattice $H^2(S, \mathbb{Z})$ of a smooth generic K3 surface such that the restriction of the intersection form is a non-degenerate bilinear form on Λ of signature $(1, \mathrm{rk}\Lambda - 1)$. For the class of threefolds X considered in this paper, Λ will be naturally identified with the second cohomology lattice of any sufficiently generic smooth K3 fiber of X. As explained in Section 2, Poincaré duality then leads to a natural identification of the lattice of vertical curve classes on X with the dual lattice $\Lambda^\vee$.

For further reference note that Λ is naturally identified with a finite index sublattice of $\Lambda^\vee$ using the given bilinear pairing. Moreover, since $\Lambda^\vee$ is contained in $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$, this bilinear pairing induces a $\mathbb{Q}$-valued pairing on $\Lambda^\vee$. The quadratic form associated to the induced bilinear pairing will be written as $d \mapsto d^2 \in \mathbb{Q}$ for any $d \in \Lambda^\vee$.

In this context, the main object of study of this work is the thermal partition function constructed by Denef and Moore and [14] for D4-D2-D0 bound states, which is reviewed in Section 2. The underlying framework for this construction is an euclidean IIA compactification on X where the time direction is periodic with period $1/T$. For simplicity the background B-field as well as the three-form RR field C_3 will be set to zero, but the background RR field C_0 will have a nontrivial constant expectation value. One also sets $\tau = C_0 + i/g_s T$ where g_s is the IIA string coupling.

The discrete topological invariants of vertical D4-D2-D0 configurations on X are triples $\gamma = (r, d, n)$ where $r, n \in \mathbb{Z}$ are D4 and anti-D0 multiplicities respectively, while $d \in \Lambda^\vee$ is the D2 charge. The BPS indices $\Omega(\gamma)$ counting such states will be denoted by $\Omega(\gamma)$. The thermal partition function can be naturally written as a finite sum

$$Z_{BPS}(X, r; \tau, \bar\tau) = \sum_{\delta \in \Lambda^\vee/r\Lambda} Z_{BPS}(X, r, \delta; \tau)\Theta^*_{r,\delta}(\tau, \bar\tau) \tag{1.1}$$

over equivalence classes $\delta \in \Lambda^\vee/r\Lambda$. For each such equivalence class,

$$Z_{BPS}(X, r, \delta; \tau) = \sum_n \Omega(\gamma)e^{2\pi i \tau(n + d^2/2r - 1)}.$$

and

$$\Theta^*_{r,\delta}(\tau, \bar\tau) = \sum_{\alpha \in \Lambda} e^{-2\pi i \tau(d + r\alpha)^2_-/2r - 2\pi i \bar\tau(d + r\alpha)^2_+/2r}.$$

where $d \in \Lambda^\vee$ is any representative of δ. $\Theta^*_{r,\delta}$ is actually the complex conjugate of a Siegel theta function of a coset of $\sqrt{r}\Lambda$. As shown in Section 2 the above series are independent of the choice of

d. Note that a similar decomposition of the thermal partition function occurs in [14, Sect. 2.2] with one significant difference. In the cases studied in loc. cit. the generic supersymmetric D4-D2-D0 configuration consists of a single D4-brane wrapping a smooth very ample divisor in X bound to some D2 and anti-D0 branes. In the present case, a generic stable configuration consists of a rank r stable vector bundle supported on a smooth K3 fiber, and cannot be deformed to a single D4-brane wrapping a smooth surface. This is reflected in the presence of the cosets $\Lambda^\vee/r\Lambda$ with $r \geq 1$ in the right hand side of (2.9), as opposed to the analogous formulas in [14, Sect. 2.2]. This will turn out to have some important arithmetic consequences.

Mathematically the BPS indices $\Omega(\gamma)$ are generalized Donaldson-Thomas invariants [54, 34, 29] counting semistable vertical sheaves on X suppported on K3 fibers. More precisely, the formalism of [34, 29] yields rational invariants $DT(\gamma) \in \mathbb{Q}$ which are conjecturally related to the integral ones $\Omega(\gamma)$ by the multicover formula

$$DT(\gamma) = \sum_{\substack{k \in \mathbb{Z}, \, k \geq 1 \\ \gamma = k\gamma'}} \frac{1}{k^2} \Omega(\gamma').$$

Some aspects of generalized Donaldson-Thomas invariants are reviewed in Section 2.3. Summing over multicovers, one obtains a partition function for rational invariants. Before writing it explicitly, note that for any $r, r' \in \mathbb{Z}$, $r, r' \geq 1$, such that $r = kr'$ with $k \in \mathbb{Z}$ there is an injective morphism

$$f_{r',k} : \Lambda^\vee/r'\Lambda \to \Lambda^\vee/r\Lambda, \qquad f_{r',k}([d]_{r'}) = [kd]_r \text{ for all } d \in \Lambda^\vee.$$

Here $[d]_s \in \Lambda^\vee/s\Lambda$ denotes the equivalence class of $d \in \Lambda^\vee$ mod $s\Lambda$ for any $s \in \mathbb{Z}$, $s \geq 1$. Then the rank $r \geq 1$ partition function for rational invariants reads

$$Z_{DT}(X, r; \tau, \bar{\tau}) = \sum_{\delta \in \Lambda^\vee/r\Lambda} Z_{DT}(X, r, \delta; \tau)\Theta^*_{r,\delta}(\tau, \bar{\tau}) \tag{1.2}$$

where

$$Z_{DT}(X, r, \delta; \tau) = \sum_{\substack{k \in \mathbb{Z}, \, k \geq 1 \\ (r,\delta) = (kr', f_{r',k}(\delta'))}} \frac{1}{k^2} Z_{BPS}(X, r', \delta'; k\tau).$$

1.2   The main formula

The main result of this paper is an explicit conjectural formula for the partition function (1.2). This formula will be written in terms of a certain vector valued modular form $\widetilde{\Phi}$ associated to an irreducible nodal K3 pencil $X \to C$ via the construction of [41, 33]. As explained in detail in Section 3, this form encodes the Noether-Lefschetz numbers of an associated smooth pencil $\widetilde{X}$ over a double cover of C. The form $\widetilde{\Phi}$ takes values in the Weil representation determined by the lattice Λ equipped with the intersection form, and has weight $11 - \text{rk}(\Lambda)/2$. Leaving the technical

details for section 3.3, note that the latter is a representation of the metaplectic cover of $SL(2, \mathbb{Z})$ on the vector space $V = \mathbb{C}[\Lambda^\vee/\Lambda]$. The components of $\widetilde{\Phi}$ with respect to the natural basis of (e_g), $g \in \Lambda^\vee/\Lambda$ will be denoted by $\widetilde{\Phi}_g$. Then the main formula reads

$$Z_{DT}(X, r, \delta; \tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z}, \ k,l \geq 1 \\ kl=r, \ \delta=f_{l,k}(\eta)}} \sum_{s=0}^{l-1} l e^{-\pi i s l \delta^2/r^2} \left(\Delta^{-1} \widetilde{\Phi}_{[\eta]_1} \right) \left(\frac{k\tau + s}{l} \right) \tag{1.3}$$

where $\Delta(\tau) = \eta(\tau)^{24}$. Note that in the right hand side $\eta \in \Lambda^\vee/l\Lambda$ is uniquely determined by δ since the morphism $f_{l,k} : \Lambda^\vee/l\Lambda \to \Lambda^\vee/r\Lambda$ is injective. Moreover $[\eta]_1 \in \Lambda^\vee/\Lambda$ denotes the image of η under the natural projection $\Lambda^\vee/l\Lambda \twoheadrightarrow \Lambda^\vee/\Lambda$.

Special cases of the above formula have been rigorously proven by Gholampour and Shehsmani [20] for K3 fibrations without singular fibers and primitive charge vectors as well as for rank $r = 1$ invariants on irreducible nodal K3 pencils. Other special cases for arbitrary $r \geq 1$ were obtained in [15] for elliptic K3 pencils in Weierstrass form using the Furier-Mukai transform.

A string theoretic derivation of equation (1.3) is given in Section 4 using adiabatic IIA/heterotic duality. The main idea is that for the purpose of vertical BPS index the K3 pencil can be regarded as a family of world sheet conformal field theories for a T^4 compactification of the $E_8 \times E_8$ heterotic string. Such an identification is not canonical because the restriction of the Calabi-Yau threefold metric to a K3 surface is not in general hyper-Kähler. Nevertheless it can be used as a very efficient tool since the BPS index of D4-D2-D0 supported on a K3 fiber is invariant under metric perturbations. The conclusion is that counting D4-D2-D0 bound states supported on nodal fibers is equivalent to counting Dabholkar-Harvey states in a smooth heterotic conformal field theory.

At the same time, a mathematical derivation of the same formula is given in Section 5 using previous results of Gholampour, Sheshmani and Toda [22]. Using recursive methods, Section 5 gives a rigorous proof of equation (1.3) provided that one assumes the Gromov-Witten/stable pair correspondence [50] to hold. This proof is based on the explicit results obtained by Maulik and Pandharipande [41] for vertical Gopakumar-Vafa invariants of K3 pencils, as well as a remarkable combinatorial identity for the Fourier coefficients of a certain meromorphic Jacobi form of negative index.

Finally, note that formula (1.3) yields the following expression for the total rank r partition function

$$Z_{DT}(X, r; \tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z}, \ k,l \geq 1 \\ kl=r}} \sum_{\eta \in \Lambda^\vee/l\Lambda} l \left(\Delta^{-1} \widetilde{\Phi}_{[\eta]_1} \right) \left(\frac{k\tau + s}{l} \right) \Theta^*_{l,\eta} \left(\frac{k\tau + s}{l}, \frac{k\bar\tau + s}{l} \right). \tag{1.4}$$

For an interesting special case of this formula, suppose the intersection form of Λ is unimodular. In that case, $\Lambda^\vee/\Lambda$ is trivial and the Noether-Lefschetz numbers are encoded in an ordinary modular

form $\widetilde{\Phi}$. Then equation (4.20) specializes to

$$
\begin{aligned}
Z_{DT}(X, r; \tau) &= \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z},\ k,l \geq 1 \\ kl = r}} l\big(\Delta^{-1}\widetilde{\Phi}\big) \left(\frac{k\tau + s}{l}\right) \Theta^* \left(\frac{k\tau + s}{l}, \frac{k\bar{\tau} + s}{l}\right) \\
&= \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z},\ k,l \geq 1 \\ kl = r}} l Z_{DT}\left(X, 1; \frac{k\tau + s}{l}\right).
\end{aligned}
\tag{1.5}
$$

where $\Theta(\tau, \bar{\tau})$ is the non-holomorphic Siegel theta function associated to the self-dual lattice. Since $Z_{DT}(X, 1; \tau)$ is a non-holomorphic modular function of weight $(-1, 1)$ (of weight minus one in τ and weight one in $\bar{\tau}$), the above formula is precisely the order r Hecke transform. At this point recall that the partition function of Vafa-Witten theory with gauge group $U(r)$ was shown in [44] to be given precisely by an order r Hecke transform of its $U(1)$ counterpart. The above formula generalizes this construction to vertical D4-D2-D0 partition functions for K3 fibered Calabi-Yau threefolds. Note also that for non-self-dual polarizing lattices the formula (1.3) is not a Hecke transform, but the partition function is still expected to be a modular function on physics grounds [14, 18]. This will be discussed next.

1.3 Modularity results

As mentioned above, physical arguments based on S-duality [14] and the M5-brane elliptic genus [18] yield precise modularity conjectures for the partition functions (1.3). Leaving the details for Section 6, the main result proved in this paper is that for fixed $r \geq 1$, the collection $\big(Z_{DT}(X, r, \delta; \tau)\big)$, $\delta \in \Lambda^\vee/r\Lambda$, determines a vector valued modular form for a certain representation of the metaplectic group $Mp(2, \mathbb{Z})$ of weight $-1 - \mathrm{rk}(\Lambda)/2$. This representation is obtained from the dual of the Weil representation by a lattice rescaling. At the same time the collection $\big(\Theta^*_{r,\delta}(\tau, \bar{\tau})\big)$, $\delta \in \Lambda^\vee/r\Lambda$, determines a non-holomorphic vector valued modular form with values in the dual representation, and weight $\mathrm{rk}(\Lambda)/2$. Therefore the full rank r partition function (1.4) is a non-holomorphic modular form of weight $(-1, 1)$, as predicted in [14, 18] on physics grounds. In order to avoid a very technical discussion at this point in the paper, the reader is referred to Section 6 for more details. However, it is worth noting that the above modularity result provides a new transformation acting on vector valued modular forms which is somewhat similar to a construction carried out in [52, Thm. 3.1]. On the other hand, given the present context, this construction could be also regarded as a generalization of the Hecke transform to vector valued modular forms in the Weil representation. An interesting open question is whether this construction is related to the existing one in the mathematical literature [9].

2   Vertical D4-D2-D0 bound states on K3 fibrations

This section will review some basic facts on vertical supersymmetric D4-D2-D0 bound states on K3 fibered Calabi-Yau threefolds. The presentation will follow closely the treatment in Sections 2.1 and 2.2 of [14], adapting their considerations to K3 fibers rather than very ample divisors. Below X will be a smooth projective Calabi-Yau threefold equipped with a projection map $\pi : X \to \mathbb{P}^1$ and a section $\sigma : \mathbb{P}^1 \to X$. The Calabi-Yau condition implies that the generic fibers of π will be smooth K3 surfaces. Singular fibers will be in general present, but throughout this paper it will be assumed that all such fibers have only isolated simple nodal points. It will be also assumed for simplicity that the integral cohomology of X is torsion free and that $h^{1,0}(X) = 0$. These assumptions are easily satisfied for sufficiently generic models.

2.1   The vertical charge lattice

For a type IIA compactification on X, the lattice of electric and magnetic charges of four dimensional BPS particles is isomorphic to the degree zero K-theory of X. In the absence of torsion, the lattice of charges can be identified with the even integral homology of X using the Chern character and Poincaré duality. In particular the charges of D4-D2-D0 bound states supported on the fibers of π take values in the sublattice

$$\mathbb{Z}\langle D\rangle \oplus H_2(X,\mathbb{Z})^\pi \oplus H_0(X,\mathbb{Z}) \subset H_{\mathsf{even}}(X,\mathbb{Z}),$$

where $D \in H_4(X,\mathbb{Z})$ is the K3 fiber class and $H_2(X,\mathbb{Z})^\pi \subset H_2(X,\mathbb{Z})$ is the kernel of the pushforward map $\pi_* : H_2(X,\mathbb{Z}) \to H_2(\mathbb{P}^1,\mathbb{Z})$.

Since one assumes the existence of a section, there are direct sum decompositions

$$H_4(X,\mathbb{Z}) \simeq \mathbb{Z}\langle D\rangle \oplus \Lambda, \qquad H_2(X,\mathbb{Z}) \simeq \mathbb{Z}\langle C\rangle \oplus H_2(X,\mathbb{Z})^\pi$$

where C, D are the section class and the K3 fiber class respectively. Moreover $\Lambda \subset H_4(X,\mathbb{Z})$ is the sublattice generated by horizontal 4-cycles relative to the map π, i.e. 4-cycles which project surjectively to $\mathbb{P}^1$. Poincaré duality yields a nondegenerate bilinear pairing

$$H_2(X,\mathbb{Z})^\pi \times \Lambda \to \mathbb{Z}$$

which identifies $H_2(X, \mathbb{Z})^\pi$ with the dual lattice $\Lambda^\vee$. In conclusion, the charges of vertical D4-D2-D0 bound states take values in a lattice of the form in

$$\Gamma = \mathbb{Z} \oplus \Lambda^\vee \oplus \mathbb{Z}.$$

A typical charge vector will be denoted by $\gamma = (r, d, n)$ where $r \geq 1$ is the D4 brane multiplicity, d is the effective D2-brane charge and n the effective anti-D0 brane charge.

Moreover, for any smooth fiber X_p, $p \in \mathbb{P}^1$, one has a natural restriction map

$$H^2(X, \mathbb{Z}) \to H^2(X_p, \mathbb{Z})$$

whose image is contained in the sublattice of $(1, 1)$ classes, $H^{1,1}(X_p) \cap H^2(X, \mathbb{Z})$. Using Poincaré duality, this yields a natural restriction map

$$\Lambda \to H^{1,1}(X_p) \cap H^2(X, \mathbb{Z}) \tag{2.1}$$

which is furthermore injective for any smooth fiber X_p. For sufficiently generic models this map is in fact an isomorphism for any sufficiently generic smooth fiber X_p. However, it fails to be surjective at special points $p \in \mathbb{P}^1$, where the rank of $H^{1,1}(X_p) \cap H^2(X, \mathbb{Z})$ jumps. This jumping phenomenon is studied in detail in the mathematics literature on Noether-Lefschetz loci [41, 33]. The results used in this paper are reviewed in detail in Section 3. For the purpose of this section, note that the map (2.1) will be assumed to be a primitive lattice embedding for all smooth fibers X_p. Moreover, note that the intersection pairing on $H^2(X_p, \mathbb{Z})$ determines by restriction a nondegenerate symmetric bilinear form on Λ. Throughout this paper this form will be assumed to be independent of the point $p \in \mathbb{P}^1$. These conditions are part of the definition of a lattice polarized family of K3 surfaces, which is reviewed in detail in Section 3. In addition one also requires the signature of the induced form on Λ to be $(1, \mathrm{rk}(\Lambda) - 1)$.

In conclusion, Λ will be endowed from this point on with a nondegenerate symmetric bilinear form which agrees with the intersection product on each smooth fiber. This form will be denoted by

$$(\alpha, \alpha') \mapsto \alpha \cdot \alpha'$$

for any $\alpha, \alpha' \in \Lambda$. By construction, this pairing is integral and even, but not unimodular. Therefore it determines an embedding $\Lambda \hookrightarrow \Lambda^\vee$ which will not be in general surjective. The quotient $G = \Lambda^\vee / \Lambda$ is a finite abelian group which will play a central part in this paper. Using this embedding Λ will be identified in the following with a sublattice of $\Lambda^\vee$. Furthermore, the symmetric nondegenerate bilinear form on Λ determines a $\mathbb{Q}$-valued symmetric nondegenerate bilinear form on $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$, hence also on $\Lambda^\vee$, which is contained in $\Lambda_\mathbb{Q}$. This form will be similarly written as

$$(d, d') \mapsto d \cdot d'$$

for any $d, d' \in \Lambda^\vee$. Note that $d \cdot \alpha \in \mathbb{Z}$ for any $d \in \Lambda^\vee$, $\alpha \in \Lambda$.

The main goal of this paper is to derive explicit formulas for the supersymmetric indices of vertical D4-D2-D0 BPS configurations of fixed charge γ, for an arbitrary fixed Kähler class ω on X. For simplicity the B-field will be set to zero.

2.2 BPS indices and partition functions

For each charge vector γ there is a Hilbert space of four dimensional BPS particles $\mathcal{H}_\omega(X, \gamma)$ which carries an action of the three dimensional rotation group $SO(3) \subset SO(1,3)$. On general symmetry grounds this space is of the form

$$\mathcal{H}_\omega(X, \gamma) \simeq 2\left((0) + (1/2)\right) \otimes \mathcal{H}_\omega^{\text{int}}(X, \gamma)$$

where $\mathcal{H}_\omega^{\text{int}}(X, \gamma)$ is obtained by quantizing the internal degrees of freedom of these particles. The BPS index $\Omega_\omega(\gamma)$ is then defined as the Witten index

$$\Omega_\omega(\gamma) = \text{Tr}_{\mathcal{H}_\omega^{\text{int}}(X, \gamma)}(-1)^{2J_3} \tag{2.2}$$

where $J_3 \subset SO(3)$ is the Cartan generator of the four dimensional little group.

Note that a simple application of the Kontsevich-Soibelman wallcrossing formula [34] shows that for vertical configurations the indices $\Omega_\omega(\gamma)$ are independent of the Kähler class ω. This follows from the observation that $(\gamma, \gamma') = 0$ for any two vertical charge vectors with respect to the natural symplectic pairing on $H^{\text{even}}(X, \mathbb{Z})$. Therefore the subscript ω will be omitted in the following.

One more useful observation is that the BPS degeneracies $\Omega(r, d, n)$ are invariant under transformations of the form

$$(r, d, n) \mapsto (r, d, n)\left(r, d + r\alpha, n - d \cdot \alpha - \frac{1}{2}r\alpha^2\right), \tag{2.3}$$

for any element $\alpha \in \Lambda$. Such transformations are associated to large radius monodromy transformations in the complexified Kähler moduli space shifting the B-field as $B \mapsto B + \alpha$.

As explained in Sections 2.1 and 2.2 of [14], there is a natural construction for generating functions of these indices derived from the natural DBI action on D4-branes. A central element in this construction is the jumping phenomenon associated to Noether-Lefschetz loci for K3 fibrations, which was briefly explained above. As opposed to the current case, the treatment of [14] is focused on a rank one D4-brane wrapping a smooth very ample divisor in X. However, their considerrations can be easily adapted to the situation at hand.

Suppose first the D4-brane rank is $r = 1$. In this case a generic supersymmetric D4-D2-D0 configuration of charge $\gamma = (1, d, n)$ will be a bound state of one D4 brane and k anti-D0 branes

supported on an arbitrary smooth fiber X_p. One also has an $U(1)$ flux $F \in H^2(X_p, \mathbb{Z})$ on the D4-brane such that

$$F \cdot \alpha = d(\alpha) \tag{2.4}$$

for any $\alpha \in \Lambda$. By supersymmetry, F has to be a $(1,1)$ class. As in [14], equation (2.4), the effective anti-D0 charge n is given by

$$n = k - \frac{F^2}{2}. \tag{2.5}$$

Recall that under the current assumptions Λ is a primitive sublattice of $H^{1,1}(X_p) \cap H^2(X_p, \mathbb{Z})$, which is in turn a primitive sublattice of $H^2(X_p, \mathbb{Z})$. Let $\Lambda^\perp \subset H^2(X_p, \mathbb{Z})$ denote the sublattice consisting of all elements which are orthogonal to Λ with respect to the natural intersection pairing. Then note that there is a direct sum decomposition

$$H^2(X_p, \mathbb{Q}) \simeq \Lambda_\mathbb{Q} \oplus \Lambda_\mathbb{Q}^\perp. \tag{2.6}$$

Let $F = F^\| + F^\perp$ be the corresponding decomposition of F. Using the natural inclusion $\Lambda^\vee \subset \Lambda_\mathbb{Q}$ equation (2.4) yields an identification $F^\| = d$.

The construction of the string theoretic generating function for the BPS degeneracies will follow closely Sections 2.1 and 2.2 of [14]. The generating function will be a thermal partition function working in an euclidean four dimensional background of the form $\mathbb{R}^3 \times S^1$, where the time direction is periodic, with period $1/T$. The background B-field and the three-form RR field C_3 will be set to zero, while the background RR field C_0 will have a nontrivial constant expectation value. As in equation (2.9) of loc. cit. the contribution of all such supersymmetric configurations supported on a fixed smooth fiber X_p to the partition function will be given by

$$\mathrm{Tr}_{\mathcal{H}^{\mathrm{int}}(X_p, 1; F, k)} (-1)^{2J_3} e^{-H/T - 2\pi i C_0 n}$$

where H is the Hamiltonian acting on the internal Hilbert space $\mathcal{H}^{\mathrm{int}}(X_p, 1; F, k)$. Evaluating this trace as in equations (2.10)-(2.14) of loc. cit. results in an expression of the form

$$c(X_p, 1, F, k) e^{2\pi i \tau (k - F_-^2/2 - \chi(X_p)/24) - 2\pi i \bar\tau F_+^2/2}$$

where

$$\tau = C_0 + \frac{i}{g_s T}, \qquad c(X_p, 1, F, k) = \mathrm{Tr}_{\mathcal{H}^{\mathrm{int}}(X_p, 1; F, k)} (-1)^{2J_3}$$

and F_+, F_- are the self-dual, respectively anti-self-dual parts of F with respect to the induced Kähler class $\omega|_{X_p}$.

Now the important observation is that $\omega|_{X_p} \in \Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$, hence one has

$$F_+ = F_+^\|, \qquad F_- = F^\perp = F_-^\perp$$

Then

$$k - F_-^2/2 = n + F_+^2/2 = n + \left(F_+^{\|}\right)^2/2 = n + \left(F^{\|}\right)^2/2 - \left(F_-^{\|}\right)^2/2,$$

where $F^{\|} = d$. Therefore, since $\chi(X_p) = 24$, one obtains

$$e^{2\pi i \tau(k - F_-^2/2 - \chi(X_p)/24) - 2\pi i \bar{\tau} F_+^2/2} = e^{2\pi i \tau(n + d^2/2 - 1)} e^{-2\pi i \tau d_-^2/2 - 2\pi i \bar{\tau} d_+^2/2}$$

which depends only on (d, n). Then, summing over (F, k), one obtains

$$\sum_{(F,k)} c(X_p, 1, F, k) e^{2\pi i \tau(k - F_-^2/2 - \chi(X_p)/24) - 2\pi i \bar{\tau} F_+^2/2} =$$

$$\sum_{d,n} \Omega(X_p, 1, d, n) e^{2\pi i \tau(n + d^2/2 - 1)} e^{-2\pi i \tau d_-^2/2 - 2\pi i \bar{\tau} d_+^2/2},$$

where

$$\Omega(X_p, 1, d, n) = \sum_{F, F^{\|} = d} c(X_p, 1, F, n + F^2/2).$$

Moreover, note that $\Omega(X_p, 1, d, n)$ and $n + d^2/2$ are invariant under transformations of the form (2.3) with $r = 1$. Therefore for any $d \in \Lambda^\vee$, the sum

$$\sum_n \Omega(X_p, 1, d, n) e^{2\pi i \tau(n + d^2/2 - 1)}$$

depends only on the equivalence class $\delta = [d]_1 \in \Lambda^\vee/\Lambda$ of $d \bmod \Lambda$. Hence it will be denoted by $Z(X_p, 1, \delta; \tau)$. At the same time, for any $d \in \Lambda^\vee$, the sum

$$\sum_{\alpha \in \Lambda} e^{-2\pi i \tau(d + \alpha)_-^2/2 - 2\pi i \bar{\tau}(d + \alpha)_+^2/2}$$

is the complex conjugate of a non-holomorphic Siegel theta function $\Theta_\delta^*(\tau, \bar{\tau})$ which also depends only on the equivalence class δ of $d \bmod \Lambda$. In conclusion for any smooth fiber X_p, the sum

$$\sum_{(F,k)} c(X_p, 1, F, k) e^{2\pi i \tau(k - F_-^2/2 - \chi(X_p)/24) - 2\pi i \bar{\tau} F_+^2/2}$$

can be written as

$$\sum_{\delta \in \Lambda^\vee/\Lambda} Z(X_p, 1, \delta; \tau) \Theta_\delta(\tau, \bar{\tau}).$$

In order to compute the full degeneracies $\Omega(\gamma)$ one has to integrate over all possible locations of the fiber X_p, including singular fibers. The mathematical framework for such a computation is the theory developed by Behrend [4] which allows one to write $\Omega(\gamma)$ as the integral of a certain constructible function on the moduli space of supersymmetric D4-D2-D0 configurations. Leaving the details for Sections 2.3 and 4.1, it suffices to note here that the effect of integrating over all

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fibers is to convert $\Omega(X_p, 1, d, n)$ into $\Omega(1, d, n)$ leaving the general form of the partition function unchanged. In conclusion the partition function for rank $r = 1$ invariants will take the form

$$Z_{BPS}(X, 1; \tau, \bar\tau) = \sum_{\delta \in \Lambda^\vee/\Lambda} Z_{BPS}(X, 1, g; \tau)\Theta_\delta^*(\tau, \bar\tau) \qquad (2.7)$$

where

$$Z_{BPS}(X, 1, g; \tau) = \sum_n \Omega(1, d, n)e^{2\pi i\tau(n+d^2/2-1)}\Big|_{[d]=g}.$$

Note that $Z_{BPS}(X, 1, \delta; \tau)$ is a power series in $q^{1/2m} = e^{\pi i\tau/m}$ for any $\delta \in \Lambda^\vee/\Lambda$, where m is the determinant of the intersection form on Λ. Moreover, according to [14], S-duality arguments imply that the collection $(Z_{BPS}(X, 1, \delta; \tau))_{\delta \in \Lambda^\vee/\Lambda}$ must form a vector-valued modular form of weight $-1-\mathrm{rk}(\Lambda)/2$ with values in the Weil representation. This implies in particular that $Z_{BPS}(X, 1; \tau, \bar\tau)$ must be a weight (-1) non-holomorphic modular form.

The next goal is to generalize the above construction to higher rank $r \geq 1$. This will be carried out in this section for primitive charge vectors $\gamma = (r, d, n)$. The general case can then be obtained by summation using the multicover formula for Donaldson-Thomas invariants, as shown in Section 4.3. For primitive charge vectors the generic supersymmetric D4-D2-D0 configuration is a D4-brane supported on a smooth fiber X_p equipped with an $U(r)$ gauge field A. The topological invariants of A will be the first Chern class

$$\beta = \mathrm{Tr}(F)$$

and the second Chern class, or instanton number,

$$k = \frac{\beta^2}{2} - \frac{\mathrm{Tr}(F^2)}{2},$$

where F is the curvature of A. Note that

$$n = k - \frac{\beta^2}{2}$$

and

$$\beta \cdot \alpha = d(\alpha)$$

for any $\alpha \in \Lambda$. Moreover, $d = \beta^\parallel$ with respect to the decomposition (2.6).

By supersymmetry the $U(r)$ gauge field must satisfy the Donaldson-Uhlenbeck-Yau equations. Therefore the field strength F of A must be a $(1, 1)$ form on X_p such that

$$\omega_p \wedge F = \lambda\omega_p^2 \qquad (2.8)$$

where $\omega_p = \omega|_{X_p}$ is the restriction of the ambient Kähler class, and $\lambda = (\omega_p \cdot \beta)/\omega_p^2$ is constant. Now let $A_0 = A - \mathrm{Tr}(A)/r$ be the traceless part of A. This is a $PU(r) \simeq SU(r)/\mu_r$ gauge field on X_p with field strength

$$F_0 = F - \frac{\beta}{r}I_r.$$

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Equation (2.8) implies that
$$\omega_p \wedge F_0 = 0,$$
that is A_0 is an antiself-dual field configuration. Hence one has
$$F_- = F_0 + \beta_-, \qquad F_+ = \beta_+.$$
Let $k_0 = -\mathrm{Tr}(F_0^2)/2$, which is a rational number in $(1/r)\mathbb{Z} \subset \mathbb{Q}$. Since F_0 is traceless, it follows that
$$n = k_0 - \frac{\beta^2}{2r}.$$

Again, the contribution of all such supersymmetric configurations to the thermal partition function will be given by
$$\mathrm{Tr}_{\mathcal{H}^{\mathrm{int}}(X_p, r; \beta, k)}(-1)^{2J_3} e^{-\beta H - 2\pi i C_0 n}.$$

Evaluating this trace by analogy with equations (2.10)-(2.14) in [14] yields in this case
$$c(X_p, r, F, k) e^{2\pi i \tau (k_0 - \beta_-^2/2r - r\chi(X_p)/24) - 2\pi i \bar{\tau} \beta_+^2/2r}.$$

By analogy with the rank one case, this expression is further equal to
$$c(X_p, r, F, k) e^{2\pi i \tau (n + d^2/2r - r) - 2\pi i \tau d_-^2/2r - 2\pi i \bar{\tau} d_+^2/2r}.$$

Again, note that the BPS index $\Omega(X_p, \gamma)$ and $n + d^2/2r$ are invariant under monodromy transformations of the form (2.3). Therefore for any $d \in \Lambda^\vee$, the sum
$$\sum_n \Omega(X_p, \gamma) e^{2\pi i \tau (n + d^2/2r - 1)}$$
depends only on the equivalence class $\delta = [d]_r \in \Lambda^\vee/r\Lambda$ of d mod $r\Lambda$. Hence it will be denoted by $Z(X_p, r, \delta; \tau)$. At the same time, for any $d \in \Lambda^\vee$, the sum
$$\sum_{\alpha \in \Lambda} e^{-2\pi i \tau (d + r\alpha)_-^2/2r - 2\pi i \bar{\tau}(d + r\alpha)_+^2/2r}$$
will also depend only on the equivalence class of d mod $r\Lambda$. Hence it will be denoted by $\Theta_{r,\delta}(\tau, \bar{\tau})$. Then one obtains a final expression for the rank r partition function of the form
$$Z_{BPS}(X, r; \tau, \bar{\tau}) = \sum_{\delta \in \Lambda^\vee/r\Lambda} Z_{BPS}(X, r, \delta; \tau) \Theta_{r,\delta}^*(\tau, \bar{\tau}) \tag{2.9}$$
where
$$Z_{BPS}(X, r, \delta; \tau) = \sum_n \Omega(\gamma) e^{2\pi i \tau (n + d^2/2r - 1)} \Big|_{[d]=\delta}.$$
Note that $Z_{BPS}(X, r, \delta; \tau)$ is a power series in $q^{1/2lr} = e^{\pi i \tau/mr}$ for any $\delta \in \Lambda^\vee/\Lambda$, where m is the determinant of the intersection form on Λ.

2.3 Mathematical approach via Donaldson-Thomas invariants

Mathematically, supersymmetric D4-D2-D0 bound states on X are Bridgeland stable objects in the derived category of X. In this paper it will be assumed that for fixed numerical invariants Bridgeland stability reduces at large radius to Gieseker-stability for coherent sheaves. Therefore a supersymmetric D4-D2-D0 configuration will be a Gieseker semistable torsion coherent sheaf E with respect to a certain polarization ω on X. For vertical D4-D2-D0 configurations E will be also required to be set theoretically supported on a finite union of K3 fibers. Using the isomorphism $H_2(X, \mathbb{Z})^\pi \simeq \Lambda^\vee$ found in the previous section, such a sheaf has numerical invariants $\gamma = (r, d, n) \in \mathbb{Z}_{\geq 1} \times \Lambda^\vee \times \mathbb{Z}$ where

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = d, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x). \tag{2.10}$$

For completeness recall Gieseker stability for pure dimension two sheaves on X. Given a real ample class ω on X, for any such nonzero sheaf E let

$$\mu_\omega(E) = \frac{\omega \cdot \mathrm{ch}_2(E)}{\omega^2 \cdot \mathrm{ch}_1(E)/2}, \qquad \nu_\omega(E) = \frac{\chi(E)}{\omega^2 \cdot \mathrm{ch}_1(E)/2}.$$

Then Gieseker (semi)stability with respect to ω is defined by the conditions

$$\mu_\omega(E') \ (\leq) \ \mu_\omega(E) \tag{2.11}$$

for any proper nonzero subsheaf $0 \subset E' \subset E$, and

$$\nu_\omega(E') \ (\leq) \ \nu_\omega(E) \tag{2.12}$$

if the slope inequality (2.11) is saturated.

For any $\gamma = (r, d, n)$ let $\mathcal{M}_\omega(\gamma)$ denote the moduli stack of ω-semistable sheaves E with discrete invariants (2.10). Let $M_\omega(\gamma)$ be its coarse moduli space parameterizing S-equivalence classes. In the absence of semistable objects, the Donaldson-Thomas invariants $DT_\omega(\gamma)$ are defined in terms of virtual cycles; a special case of [54]. This is for example the case if γ is primitive. Moreover in this case, there exists a constructible function $\nu : M_\omega(\gamma) \to \mathbb{Z}$ constructed by Behrend [4] such that

$$DT_\omega(\gamma) = \chi(M_\omega(\gamma), \nu). \tag{2.13}$$

By definition, given a constructible function $\phi : M \to \mathbb{Z}$ on any scheme of finite type, the weighted Euler characteristic $\chi(M, \phi)$ is defined by

$$\chi(M, \phi) = \sum_{n \in \mathbb{Z}} n\chi(\phi^{-1}(n)),$$

where $\chi(\ )$ denotes the topological Euler characteristic.

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The value of ν at a point $\mathfrak{m} \in M_\omega(\gamma)$ is determined by the local scheme structure of the moduli space near $\mathfrak{m}$. Acoording to [7], the moduli space admits a local presentation as a critical locus of a polynomial function W defined on the space of infinitesimal deformations of the corresponding sheaf E on X. Then

$$\nu(\mathfrak{m}) = (-1)^d (1 - \chi(M_\mathfrak{m}(W)) \tag{2.14}$$

where d is the dimension of the space of infinitesimal deformations and $M_\mathfrak{m}(W)$ is the Milnor fiber of W at $\mathfrak{m}$. Note that if the moduli space is smooth at $\mathfrak{m}$,

$$\nu(\mathfrak{m}) = (-1)^{\dim(\mathrm{T}_\mathfrak{m} \mathrm{M}_\omega(\gamma))}. \tag{2.15}$$

In particular, if the moduli space is a smooth projective variety, the constructible function ν takes the constant value $(-1)^{\dim(\mathcal{M}_\omega(\gamma))}$, hence

$$\chi(\mathcal{M}_\omega(\gamma), \nu) = (-1)^{\dim(\mathcal{M}_\omega(\gamma))} \chi(\mathcal{M}_\omega(\gamma)).$$

From a physical perspective, the infinitesimal deformations of E are associated to complex chiral superfields in the low energy effective action of the corresponding D4-D2-D0 configuration, and W is a superpotential interaction. The moduli space is locally identified with the critical locus of W. For an isolated critical point, Milnor's result [43] shows that $\nu(\mathfrak{m})$ is the same as the dimension of the chiral ring of W. Formula (2.14) provides a suitable generalization for non-isolated vacua.

In the presence of semistable objects, one has to employ the formalism of Kontsevich and Soibleman [34] or Joyce and Song [29] to construct generalized Donaldson-Thomas invariants $DT_\omega(\gamma) \in \mathbb{Q}$. For sufficiently generic ω there are conjectural integral invariants $\Omega_\omega(\gamma) \in \mathbb{Z}$ related to the rational ones by the multicover formula

$$DT_\omega(\gamma) = \sum_{\substack{k \in \mathbb{Z},\, k \geq 1 \\ \gamma = k\gamma'}} \frac{1}{k^2} \Omega_\omega(\gamma'). \tag{2.16}$$

According to [16, 17] the integral invariants $\Omega_\omega(\gamma)$ are mathematical avatars of the BPS indices defined in (2.2). This justifies using the same notation in both cases. Moreover, as noted in the previous section, the wallcrossing formulas of [34, 29] show that these invariants are in fact independent of the Kähler class, therefore the subscript ω can be omitted.

Finally, note that the monodromy transformations (2.3) are obtained by taking a tensor product by a line bundle L on X. More precisely note that for any vertical sheaf E,

$$\mathrm{ch}_1(E \otimes_X L) = \mathrm{ch}_1(E), \qquad \mathrm{ch}_2(E \otimes_X L) = \mathrm{ch}_2(E) + c_1(L) \cdot \mathrm{ch}_1(E),$$

$$\mathrm{ch}_3(E \otimes_X L) = \mathrm{ch}_3(E) + c_1(L) \cdot \mathrm{ch}_2(E) + \frac{1}{2} c_1(L)^2 \cdot \mathrm{ch}_1(E).$$

For $c_1(L) = \alpha$, the numerical invariants of E change according to equation (2.3). Taking a tensor product as above yields an isomorphism of moduli spaces, hence it preserves Donaldson-Thomas invariants.

The plan for the rest of the paper is to provide two derivations for the expression (1.3) encoding all the above invariants. The first is a string theoretic derivation based on adiabatic IIA/heterotic duality while the second is based on the mathematical results of [22]. Both derivations rely heavily on a detailed understanding of lattice polarizations and Noether-Lefschetz loci, which is the subject of the next section.

3   Lattice polarization and Noether-Lefschetz loci

This section is a review of lattice polarized K3 fibrations and Noether-Lefshetz numbers mainly following [41, 33]. The presentation will be fairly technical by neccessity, since it lies the groundwork for the following sections.

Let $\pi : X \to \mathbb{P}^1$ be a K3 fibered smooth projective Calabi-Yau threefold with a section $\sigma : \mathbb{P}^1 \to X$ satisfying the following conditions:

a) All K3 fibers are irreducible, reduced. The generic fiber is smooth and there are finitely many singular fibers, each of them with exactly one simple node. In order to simplify the presentation, it will also be assumed that the number of singular fibers is even, although this is not an essential assumption. All the following considerations extend with minor modifications to the fibrations with odd numbers of singular fibers.

Let $S^\pi \subset \mathbb{P}^1$ be the set of critical values of π. Under the above assumptions, S^π is a finite set consisting of an even number of points. Let $f : \Sigma \to \mathbb{P}^1$ be a smooth generic double cover with branch locus S^π. Hence Σ is a hyperelliptic curve of genus

$$g(\Sigma) = |S^\pi|/2 - 1.$$

Note that there is a unique ramification point of f mapping to $\sigma \in S^\pi$. Abusing notation, it will be denoted by σ as well, the distinction being clear from the context. The set of ramification points of f will be denoted by $R^f \subset \Sigma$. Note also that if $|S^\pi|$ is odd one has to choose an extra point $\sigma_\infty \in \mathbb{P}^1$ parameterizing a generic smooth fiber, and consider a double cover with brach locus $S^\pi \cup \{\sigma_\infty\}$. All the following considerations will go through with minor modifications.

Let $X' = X \times_{\mathbb{P}^1} \Sigma$. Then X' is a singular threefold with finitely many ordinary double points corresponding to the nodal points in the fibers of π. Under the current assumption there is one nodal point $x'_\sigma \in X'$ for each $\sigma \in S^\pi$. Let $\widetilde{X} \to X$ be a small crepant resolution of singularities. Let $\tilde{\pi} : \widetilde{X} \to \Sigma$ and $\tilde{p} : \widetilde{X} \to X$ be the natural projections. The exceptional locus consists of

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finitely many disjoint $(-1, -1)$ curves $\widetilde{C}_\sigma$ on $\widetilde{X}$, in one-to-one correspondence with points $\sigma \in S^\pi$. Each such curve is at the same time a (-2)-curve on the fiber $\widetilde{X}_\sigma = \tilde{\pi}^{-1}(\sigma)$, which is an embedded resolution of the nodal surface X_σ.

Let U denote the Lorentzian rank two lattice generated by two null vectors vectors u, u^* with $u \cdot u^* = 1$. Let $\Lambda_{K3} \simeq U^{\oplus 3} \oplus \Lambda_{E_8}(-1)^{\oplus 2}$ be the middle homology lattice of a smooth generic K3 surface, where the $\Lambda_{E_8}(-1)$ denotes the E_8 root lattice equipped with a bilinear pairing given by the negative of the Cartan form. Let $\Lambda \subset \Lambda_{K3}$ be a rank $\ell \geq 1$ sublattice of signature $(1, \ell - 1)$ and $(v_1, \ldots, v_\ell)$ be an integral basis of Λ. Let also $\Lambda^\vee$ be the dual lattice and $(\check{v}^i)$, $1 \leq i \leq \ell$ be the dual basis with respect to (v_i).

The pencil $\pi : X \to \mathbb{P}^1$ will be assumed to satifsy the following additional conditions, which are easily satisfied for generic complete intersections in toric varieties.

(b) There exists a collection of divisor classes $H_1, \ldots, H_\ell \in \mathrm{Pic}(X)$, $m \geq 1$ such that the data $\left(\widetilde{X} \to \Sigma, \tilde{p}^* H_1, \ldots, \tilde{p}^* H_\ell\right)$ is a family of Λ-polarized K3 surfaces as defined in [33, Sect. 0.2.1]. In particular for any closed point $s \in \Sigma$ there is a primitive embedding $\Lambda \hookrightarrow \mathrm{Pic}(\widetilde{X}_s)$ mapping v_i to $\widetilde{H}_{i,s} = \tilde{p}^* H_i|_{\widetilde{X}_s}$ for all $1 \leq i \leq \ell$. One also requires the existence of an element $\lambda \in \Lambda$ which is mapped to a quasi-polarization of $\widetilde{X}_s$ for each $s \in \Sigma$.

A stronger condition will be assumed here, namely that $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ is freely generated by $H_1, \ldots, H_\ell$ and the K3 fiber class D. Moreover, there is a relative ample class on X over $\mathbb{P}^1$ which restricts to λ on each smooth fiber of π.

(c) For each point $\sigma \in S^\pi$ there is an orthogonal decomposition

$$\mathrm{Pic}(\widetilde{X}_\sigma) \simeq \Lambda \oplus \mathbb{Z}\langle \widetilde{C}_\sigma \rangle \tag{3.1}$$

with respect to the intersection product.

(d) For any sufficiently generic point $s \in \Sigma \setminus R^f$ the primitive embedding $\Lambda \hookrightarrow \mathrm{Pic}(\widetilde{X}_s)$ is an isomorphism.

3.1 Noether-Lefschetz numbers

Next recall the definition of Noether-Lefschetz numbers for the family $\tilde{\pi} : \widetilde{X} \to \Sigma$. Let $h \in \mathbb{Z}$ and $d = \sum_{i=1}^\ell d_i \check{v}^i \in \Lambda^\vee$. Since the bases (v_i), $(\check{v}^i)$ are fixed, d will be often be written as $d = (d_i)_{1 \leq i \leq \ell}$. Informally the Noether Lefschetz number $\widetilde{NL}_{h,d} \subset \Sigma$ is the number of points $s \in \Sigma$, counted with multiplicity, such that there exists a divisor class $\beta \in \mathrm{Pic}(\widetilde{X}_s)$ satisfying

$$\beta^2 = 2h - 2, \qquad \beta \cdot \widetilde{H}_{i,s} = d_i, \quad 1 \leq i \leq \ell. \tag{3.2}$$

A rigorous definition of Noether-Lefschetz numbers involves excess intersection theory, as shown in [41, Sect. 1.4] and [33, Sect. 0.2.2, 0.2.3]. Following [41, Sect. 1.4], consider the local system

$\widetilde{\mathcal{V}} = R^2\tilde{\pi}_*\mathbb{Z}$ on Σ and let $h : \widetilde{\mathcal{H}} \to \Sigma$ be the $\tilde{\pi}$-relative moduli space of Hodge structures of type $(1, 20, 1)$ on $\widetilde{\mathcal{V}}\otimes_{\mathbb{Z}}\mathbb{C}$. For any pair $(h, d) \in \mathbb{Z}\times\mathbb{Z}^{\ell}$ there exists a countable union of divisors $\mathcal{D}_{h,d} \subset \widetilde{\mathcal{H}}$ parameterizing Hodge structures which contain a class $\beta \in \widetilde{\mathcal{V}}_s$ satisfying conditions (3.2). One also has a canonical section $\phi : \Sigma \to \widetilde{\mathcal{H}}$ such that $\phi(s) = [H^0(\widetilde{X}_s, \mathbb{C})] \in \widetilde{\mathcal{H}}_s$ for any $s \in \Sigma$. Then

$$\widetilde{NL}_{h,d} = \int_{\Sigma} \phi^*\mathcal{D}_{h,d}. \tag{3.3}$$

According to [41, Prop. 1] the right hand side of equation (3.3) is finite although $\mathcal{D}_{h,d}$ may have infinitely many components. The proof of [41, Prop. 1] shows that the image $\phi(\Sigma)$ intersects only finitely many of them.

3.2   Local systems and jump loci

Given a pair $(h, d) \in \mathbb{Z}\times\Lambda^{\vee}$ and a point $s \in \Sigma$ let $\mathcal{B}_s^{\tilde{\pi}}(h, d)$ denote the set of classes $\beta \in \mathrm{Pic}(\widetilde{X}_s)$ satisfying conditions (3.2). This is a finite set by [41, Prop. 1]. The union $\mathcal{B}^{\tilde{\pi}}(h, d) = \cup_{s\in\Sigma}\mathcal{B}_s^{\tilde{\pi}}(h, d) \subset \widetilde{\mathcal{V}}$ decomposes as

$$\mathcal{B}^{\tilde{\pi}}(h, d) = \widetilde{\mathcal{B}}^{\mathsf{iso}}(h, d) \cup \widetilde{\mathcal{B}}^{\infty}(h, d) \tag{3.4}$$

where $\widetilde{\mathcal{B}}^{\mathsf{iso}}(h, d)$ projects to a finite subset of Σ, while $\widetilde{\mathcal{B}}^{\infty}(h, d) \subset \widetilde{\mathcal{V}}$ is a local system over Σ. For any pair (h, d) let $J_{h,d}^{\tilde{\pi}} \subset \Sigma$ be the projection of $\widetilde{\mathcal{B}}^{\mathsf{iso}}(h, d)$ to Σ. This finite subset of Σ will be called the jump locus of type (h, d). The Noether-Lefschetz numbers decompose accordingly as

$$\widetilde{NL}_{h,d} = NL_{h,d}^{\mathsf{iso}} + NL_{h,d}^{\infty}. \tag{3.5}$$

The first term in the right hand side of (3.5) is a finite sum of the form

$$NL_{h,d}^{\mathsf{iso}} = \sum_{s\in J_{h,d}^{\tilde{\pi}}} \sum_{\beta\in\widetilde{\mathcal{B}}_s^{\mathsf{iso}}(h,d)} \mu(h, d, \beta) \tag{3.6}$$

where $\mu(h, d, \beta) \in \mathbb{Z}$ is the intersection multiplicity of the section $\phi(\Sigma)$ with $\mathcal{D}(h, d)$ at the closed point corresponding to β.

By definition, the second term in the right hand side of (3.5) is computed as follows. Note that there is a line bundle $\mathcal{K} = R^0\tilde{\pi}_*\omega_{\tilde{\pi}}$ on Σ, where $\omega_{\tilde{\pi}}$ is the relative dualizing sheaf. Then

$$NL_{h,d}^{\infty} = -\int_{\widetilde{\mathcal{B}}^{\infty}(h,d)} c_1(\mathcal{K}). \tag{3.7}$$

Using the natural inclusion $\Lambda \hookrightarrow \Lambda^{\vee}$ determined by the intersection form, condition (d) in this section implies that

$$\widetilde{\mathcal{B}}_s^{\infty}(h, d) = \begin{cases} \{\alpha\} \subset \Lambda, & \text{if } d = \alpha \text{ for some } \alpha \in \Lambda \text{ and } h = 1 + \alpha^2/2, \\ \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.8}$$

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for any $s \in \Sigma$. Therefore $\mathcal{B}^{\infty}(h, d)$ is either empty or a rank one local system on Σ. Since X is K-trivial, one then obtains

$$NL_{h,d}^{\infty} = \begin{cases} -4, & \text{if } d = \alpha \text{ for some } \alpha \in \Lambda \text{ and } h = 1 + \alpha^2/2, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

For future reference consider the following example. Let $\sigma \in \Sigma$ be a ramification point of f. Using the isomorphism (3.1), any class $\beta \in \mathrm{Pic}(\widetilde{X}_\sigma)$ is written as

$$\beta = \alpha + l\widetilde{C}_\sigma$$

with $l \in \mathbb{Z}$ and $\alpha \in \Lambda$. This implies that $\beta \cdot \widetilde{H}_{i,\sigma} = \alpha \cdot \widetilde{H}_{i,\sigma}$ for $1 \le i \le \ell$ and $\beta^2 = \alpha^2 - 2l^2$. Therefore, under the current assumptions, for any ramification point $\sigma \in R^f$, the component $\widetilde{\mathcal{B}}_\sigma^{\mathsf{iso}}(h, d)$ is empty unless

$$(h, d) = (1 + \alpha^2/2 - l^2, \alpha)$$

for some $\alpha \in \Lambda$, $l \in \mathbb{Z} \setminus \{0\}$, in which case

$$\widetilde{\mathcal{B}}_\sigma^{\mathsf{iso}}(1 + \alpha^2/2 - l^2, \alpha) \simeq \{\alpha - lC_\sigma, \alpha + lC_\sigma\}.$$

Furthermore, [33, Lemma 2], implies that

$$\mu(1 - l^2, 0, \pm lC_\sigma) = 2 \tag{3.10}$$

for any $\sigma \in R^f$, $l \in \mathbb{Z} \setminus \{0\}$. More generally, by analogy with loc. cit., using condition (d) it can also be proved that

$$\mu(1 + \alpha^2/2 - l^2, \alpha \pm lC_\sigma) = 2 \tag{3.11}$$

for any $\sigma \in R^f$, $\alpha \in \Lambda$, $l \in \mathbb{Z} \setminus \{0\}$.

One can similarly define local systems and jump loci for the restriction of the family $\pi : X \to \mathbb{P}^1$ to the open subset $U^\pi = \mathbb{P}^1 \setminus S^\pi$. For any $p \in U^\pi$ and any $(h, d) \in \mathbb{Z} \times \Lambda^\vee$ let $\mathcal{B}_p^\pi(h, d)$ be the subset of classes $\beta \in \mathrm{Pic}(X_p)$ such that

$$\beta^2 = 2h - d, \qquad \beta \cdot H_i|_{X_p} = d_i, \quad 1 \le i \le \ell. \tag{3.12}$$

By construction, $\mathcal{B}_p^\pi(h, d) \simeq \mathcal{B}_{s_1}^{\tilde{\pi}}(m, h, d) \simeq \mathcal{B}_{s_2}^{\tilde{\pi}}(m, h, d)$ for any $p \in U^\pi$ and any (h, d), where $f^{-1}(p) = \{s_1, s_2\} \subset \Sigma$. In particular all $\mathcal{B}_p^\pi(h, d)$ are finite and one has again a decomposition

$$\mathcal{B}^\pi(h, d) = \mathcal{B}^{\mathsf{iso}}(h, d) \cup \mathcal{B}^{\infty}(h, d)$$

analogous to (3.4). Clearly, $\mathcal{B}_p^{\mathsf{iso}}(h, d) \simeq \mathcal{B}_{s_1}^{\mathsf{iso}}(h, d) \simeq \mathcal{B}_{s_2}^{\mathsf{iso}}(h, d)$ and $\mathcal{B}_p^{\infty}(h, d) \simeq \mathcal{B}_{s_1}^{\infty}(h, d) \simeq \mathcal{B}_{s_2}^{\infty}(h, d)$. Again, the jump locus $J_{h,d}^\pi \subset U^\pi$ is the projection of $\mathcal{B}^{\mathsf{iso}}(h, d)$. Obviously, $J^{\tilde{\pi}}{}_{h,d} \setminus R^f = f^{-1}(J_{h,d}^\pi)$.

Moreover, one can again define the intersection multiplicity $\mu(h, d, \beta)$ for any $\beta \in \mathcal{B}^{\text{iso}}(h, d)$. This will coincide with the multiplicity of the corresponding classes $\beta_i \in \mathcal{B}^{\text{iso}}_{s_i}(h, d)$, $1 \leq i \leq 2$.

Then equation (3.11) implies that

$$
NL^{\text{iso}}_{h,d} = \begin{cases} 2\sum_{p \in J_{h,d}(X)} \sum_{\beta \in \mathcal{B}^{\text{iso}}_p(h,d)} \mu(h, d, \beta) + 2|S^\pi|, & \text{if } d = \alpha \text{ for some } \alpha \in \Lambda \text{ and } h = 1 + \alpha^2/2, \\ \\ 2\sum_{p \in J_{h,\alpha}(X)} \sum_{\beta \in \mathcal{B}^{\text{iso}}_p(h,\alpha)} \mu(h, d, \beta) & \text{otherwise.} \end{cases}
$$

(3.13)

3.3 Noether-Lefschetz numbers and modular forms

According to [5, 35] any smooth lattice polarized K3 pencil determines a vector valued modular form which encodes all its Noether-Lefschetz numbers. This is briefly reviewed in [33, Sect. 0.2.4].

Let $m = |\det(M)|$, where $M_{ij} = v_i \cdot v_j$ is the intersection matrix of the basis of Λ. Let $G_1 = \Lambda^\vee/\Lambda$, where the injection $\Lambda \hookrightarrow \Lambda^\vee$ is determined by the intersection form. For any $d \in \Lambda^\vee$ let $\delta = [d]_1$ denote its equivalence class in G_1. Note that there is a $\mathbb{Q}$-valued dual pairing $\Lambda^\vee \times \Lambda^\vee \to \mathbb{Z}$ defined by

$$
\check{v}^i \cdot \check{v}^j = M^{-1}_{ij}.
$$

(3.14)

This pairing determines a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\ ,\ ) : G_1 \times G_1 \to \mathbb{Q}/\mathbb{Z}$,

$$
([d]_1, [d']_1) = d \cdot d' \bmod \mathbb{Z}.
$$

The associated quadratic form will be denoted by $\theta_1 : G_1 \to \mathbb{Q}/\mathbb{Z}$. Hence $\theta_1(\delta) = d^2/2 \bmod \mathbb{Z}$ for any $d \in \Lambda^\vee$ such that $[d]_1 = \delta$.

Finally, note that there is a canonical representation $\rho_\Lambda : Mp(2, \mathbb{Z}) \to \text{End}(\mathbb{C}[G])$ constructed by Weil [63], where $Mp(2, \mathbb{Z})$ is the metaplectic double cover of $SL(2, \mathbb{Z})$. In more detail, recall that $Mp(2, \mathbb{Z})$ is the double cover of $SL(2, \mathbb{Z})$ consisting of pairs

$$
(\sigma, \sqrt{c\tau + d}), \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})
$$

where τ is an arbitrary complex number in the upper half plane. This group does not depend on τ and is in fact generated by

$$
\widetilde{T} = (T, 1), \qquad \widetilde{S} = (S, e^{\pi i/4}),
$$

which satisfy the relations $(\widetilde{S})^2 = (\widetilde{S}\widetilde{T})^3 = (-I_2, i)$. Then the representation ρ_Λ is defined by

$$
\rho_\Lambda(\widetilde{T})(e_g) = e^{2\pi i\theta(g)} e_g, \qquad \rho_\Lambda(\widetilde{S})(e_g) = \frac{e^{\pi i(\ell-2)/4}}{\sqrt{m}} \sum_{g' \in G} e^{-2\pi i g \cdot g'} e_{g'}.
$$

(3.15)

For each pair $(h, d) \in \mathbb{Z} \times \Lambda^\vee$, $d = (d_1, \ldots, d_\ell)$, let

$$\Delta(h, d) = (-1)^\ell \det \begin{pmatrix} M & d^t \\ d & 2h - 2 \end{pmatrix}.$$

Note that

$$\frac{\Delta(h, d)}{2m} = 1 + \frac{d^2}{2} - h.$$

Then there is a vector valued modular form

$$\widetilde{\Phi}(q) = \sum_{\delta \in G} \widetilde{\Phi}_\delta(q) e_\delta \in \mathbb{C}[[q^{1/2m}]] \otimes \mathbb{C}[G]$$

of weight $w = (22 - \ell)/2$ and type ρ_Λ^* such that

$$\widetilde{NL}_{h,d} = \widetilde{\Phi}_\delta \left[\Delta(h, d)/2m \right]. \tag{3.16}$$

where $[d]_1 = \pm\delta$. Here $\Psi[s]$, $s \in (1/2m)\mathbb{Z} \subset \mathbb{Q}$, $s \geq 0$ are the Fourier coefficients of the series $\Psi(q) \in \mathbb{C}[[q^{1/2m}]]$, that is

$$\Psi(q) = \sum_{\substack{s \in (1/2m)\mathbb{Z}, \\ s \geq 0}} \Psi[s] q^s.$$

In particular, one must have

$$h \leq \frac{d^2}{2} + 1$$

for any class $\beta \in \mathcal{B}^{\tilde{\pi}}(h, d)$. This follows directly from the algebraic Hodge theorem.

4 Vertical BPS indices from adiabatic IIA/heterotic duality

This section consists of a string theoretic derivation of the main formula (1.3) from adiabatic IIA/heterotic duality for K3 fibrations.

4.1 Primitive charge vectors

First suppose $\gamma = (r, d, n)$ is primitive. In this case all semistable vertical sheaves with invariants γ are stable. According to [20, Lemma 3.1], any such sheaf E must be the extension by zero of a stable torsion sheaf F on a reduced fiber X_p of π. For any $p \in \mathbb{P}^1$ let $M_p(\gamma)$ denote the closed subspace of the coarse moduli space parameterizing isomorphism classes of stable sheaves E supported on X_p. Recall that $U^\pi = \mathbb{P}^1 \setminus S^\pi$ is the open subset parameterizing smooth fibers. Let $M_{U^\pi}(\gamma)$ denote the open subset of the moduli space parameterizing stable sheaves E supported on X_p with $p \in U^\pi$. Then using equation (2.13) one has

$$\Omega(\gamma) = \chi(M_{U^\pi}(\gamma), \nu) + \sum_{\sigma \in S^\pi} \chi(M_\sigma(\gamma), \nu). \tag{4.1}$$

The first term in the right hand side of equation (4.1) can be explicitly evaluated using the results of [20]. To explain this in some detail, let X_p be a smooth fiber of π and $\iota_p : X_p \hookrightarrow X$ denote the natural emebdding. Then any stable sheaf E suported on X_p is the extension by zero, $E \simeq \iota_{p*}(F)$, of an $\omega|_{X_p}$-stable sheaf F on X_p. The numerical invariants of F are related to those of E by the Grothendieck-Riemann-Roch formula:

$$\mathrm{rk}(F) = r, \qquad \iota_* \beta = \mathrm{ch}_2(E), \qquad k - \frac{\beta^2}{2} = n, \tag{4.2}$$

where $\beta = c_1(F)$ and $k = \int_{X_p} c_2(F)$. This implies that $M_p(\gamma)$ has finitely many disjoint components $M_{p,\beta}(\gamma)$ in one-to-one correspondence with classes $\beta \in \mathrm{Pic}(X)$ such that $\iota_* \beta = \mathrm{ch}_2(E)$. The set of all such classes β coincides with the set $\mathcal{B}_p^\pi(h, d)$ introduced in Section 3.2, where $h = 1 + \beta^2/2$.

For each $\beta \in \mathcal{B}_p^\pi(h, d)$, $k \in \mathbb{Z}$, let $M(X_p, r, \beta, k)$ be the moduli space of $\omega|_{X_p}$-stable torsion free sheaves on X_p with numerical invariants (r, β, k), where $k = n + \beta^2/2$. This is smooth and projective, of dimension

$$\dim M(X_p, r, \beta, k) = 2\left(rk - (r-1)\beta^2/2 - r^2 + 1\right) = 2(rn - r^2 + h). \tag{4.3}$$

Then one can easily construct a closed embedding $M(X_p, r, \beta, k) \hookrightarrow M_{p,\beta}(\gamma)$ which yields an isomorphism between the two sets of closed points. This implies that the reduced scheme $M_{p,\beta}(\gamma)^{\mathsf{red}}$ is isomorphic to $M(X_p, r, \beta, k)$, hence $M_{p,\beta}(\gamma)$ has the same dimension as $M(X_p, r, \beta, k)$. Hence $M_{p,\beta}$ is nonempty if and only if

$$h \geq r(r - n). \tag{4.4}$$

In fact for primitive invariants (r, β, k), the moduli space $M(X_p, r, \beta, k)$ is a smooth deformation of a Hilbert scheme of points $H^{\dim(M(X_p, r, \beta, k))/2}(S)$ on a smooth algebraic K3 surface S.

Note however that in general $M_{p,\beta}(\gamma)$ will not be isomorphic to $M(X_p, r, \beta, k)$ as a scheme since its structure sheaf can in principle contain nilpotent elements. The different scheme structure of $M_{p,\beta}(\gamma)$ will lead to nontrivial values of the Behrend function, as explained below equation (4.10). To summarize, one has a decomposition

$$\chi(M_p(\gamma), \nu) = \sum_{\substack{h \in \mathbb{Z} \\ h \geq r(r-n)}} \sum_{\beta \in \mathcal{B}_p^\pi(h,d)} \chi(M_{p,\beta}(\gamma), \nu), \tag{4.5}$$

where the sum in the right hand side is finite by the finiteness properties of the Noether-Lefschetz loci.

Now recall that for any $p \in U^\pi = \mathbb{P}^1 \setminus S^\pi$ the set $\mathcal{B}_p^\pi(h, d)$ decomposes as $\mathcal{B}_p^\pi(h, d) = \mathcal{B}_p^\infty(h, d) \cup \mathcal{B}_p^{\mathsf{iso}}(h, d)$. Note that $M_{p,\beta}(\gamma)$ is an isolated closed component of the moduli space for each $\beta \in \mathcal{B}_p^{\mathsf{iso}}(h, d)$. Using the results of [4], its contribution to the right hand side of (4.5) follows from [20, Thm. 3.18],

$$\chi(M_{p,\beta}(\gamma), \nu) = \mu(h, d, \beta)\chi(M(X_p, r, \beta, k)) = \mu(h, d, \beta)c(r(n - r) + h)), \tag{4.6}$$

22

where

$$c(r(n-r)+h) = \begin{cases} \chi(H^{r(n-r)+h}(S)), & \text{if } h \geq r(r-n), \\ 0 & \text{otherwise.} \end{cases}$$

In the above formula $H^k(S)$ denotes the Hilbert scheme of k points on a smooth generic algebraic K3 surface S. The coefficients $c(r(n-r)+h)$ are determined by Goettsche's formula [23] applied to K3 surfaces,

$$q^{-1} \sum_{k=0}^{\infty} \chi(H^k(S))q^k = \frac{1}{\eta(q)^{24}}. \tag{4.7}$$

The coefficient $\mu(h, d, \beta)$ is the same as the contribution of the isolated class β to the Noether-Lefschetz number in (3.6).

Furthermore, according to condition (e) in Section 3.2,

$$\mathcal{B}_p^{\infty}(h, d) = \begin{cases} \mathcal{B}_p^{\infty}(\alpha) = \{\alpha\} \subset \Lambda, & \text{if } (h, d) = (h(\alpha), \alpha) \text{ for some } \alpha \in \Lambda, \\ \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $h(\alpha) = 1 + \alpha^2/2$. Therefore, if $d \neq \alpha$ for some $\alpha \in \Lambda$, the moduli space $M_{U^{\pi}}(\gamma)$ will be a finite union of isolated closed components whose contributions are given by (4.6). If $d = \alpha$ for some $\alpha \in \Lambda$, according to [20, Lemma 3.7], there is a smooth connected component $M_{U^{\pi}}^{\infty}(\gamma)$ of $M_{U^{\pi}}(\gamma)$ whose set of closed points coincides with the union $\cup_{p \in U^{\pi}} M_{p,\alpha}(\gamma)$. Moreover for each $p \in U$, there is an isomorphism $M_{p,\alpha}(\gamma) \simeq M(X_p, r, \alpha, k)$, with $k = n + \alpha^2/2$. Since $M(X_p, r, \alpha, k)$ is smooth and projective of dimension (4.3), this implies that

$$\chi(M_{p,\alpha}(\gamma), \nu) = -\chi(M_{p,\alpha}(\gamma)) = -c(r(n-r)+h(\alpha)) \tag{4.8}$$

and

$$\chi(M_U^{\infty}(r, \alpha, n), \nu) = -\chi(U^{\pi})c(r(n-r)+h(\alpha)). \tag{4.9}$$

Next recall that for any $d \in \Lambda^{\vee}$, and any curve class $\beta \in \mathcal{B}^{\text{iso}}(h, d)$ one has $h \leq d^2/2 + 1$ from the algebraic Hodge theorem. Therefore, for any $\gamma = (r, d, n)$, the contribution of $M_{U^{\pi}}(\gamma)$ to the Donaldson-Thomas invariant $\Omega(\gamma)$ is

$$\chi(M_{U^{\pi}}(\gamma), \nu) = -\chi(U^{\pi})c(r(n-r)+d^2/2+1)\delta_{d,\Lambda} + \sum_{\substack{h \in \mathbb{Z} \\ r(r-n) \leq h \leq d^2/2+1}} \sum_{\beta \in \mathcal{B}^{\text{iso}}(h,\alpha)} \mu(h, d, \beta)c(r(n-r)+h) \tag{4.10}$$

where

$$\delta_{d,\Lambda} = \begin{cases} 1 & \text{if } d \in \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

For completeness note that the weights $\mu(h, d, \beta)$ in equation (4.6) have a clear physical interpretation. This was first observed in a similar context in [14, App. G]. Namely, one can easily check

that any vertical stable D4-D2-D0 configuration has exactly one normal infinitesimal deformation corresponding to translations along the base of the K3 fibration. More precisely, given a stable vertical sheaf $E = \iota_{p*}(F)$ supported on a reduced K3 fiber X_p one can easily check that the space $\mathrm{Ext}^1_X(E, E)$ of infinitesimal deformations splits as

$$\mathrm{Ext}^1_X(E, E) \simeq \mathrm{Ext}^1_{X_p}(F, F) \oplus \mathrm{Ext}^0_{X_p}(F, F).$$

The first summand parameterizes infinitesimal deformations of F as a sheaf on X_p, while the second parameterizes normal deformations in the Calabi-Yau threefold X. Moreover, stability implies that $\mathrm{Ext}^0_{X_p}(F, F) \simeq \mathbb{C}$ is one dimensional. This means that the low energy effective action of the corresponding D4-D2-D0 configuration will contain a complex chiral fields $\mathcal{X}_1, \ldots, \mathcal{X}_{d-1}$ associated to tangent fluctuations and an additional chiral field Φ associated to normal fluctuations to the fiber. Here $d = \dim \mathrm{Ext}^1_X(E, E) \geq 1$. Since the moduli space $M(X_p, r, \beta, k)$ is smooth, the tangent deformations parameterized by $\mathcal{X}_1, \ldots, \mathcal{X}_{d-1}$ are unobstructed. However, if $\beta = c_1(F)$ is an isolated curve class on X_p, the normal deformations of E will be obstructed. This will be encoded in a superpotential interaction $W(\mathcal{X}_1, \ldots, \mathcal{X}_d, \Phi) = a\Phi^l + \ldots$ where the right hand side is a polynomial of Φ and $l \geq 1$ is the degree of the lowest degree term. Then using equation (2.14), the value of the Behrend function at the point $[E]$ of the moduli space will be

$$\nu([E]) = l - 1,$$

which is the same as the dimension of the chiral ring of a Landau-Ginzburg model with superpotential $a\Phi^l + \ldots$. The vacuum D4-D2-D0 configuration is locally given by the critical points of W i.e. $\Phi^{l-1} = 0$. Hence the local ring of the component $M_{p,\beta}(\gamma)$ at $[E]$ is a nilpotent extension of the local ring of $M(X_p, r, \beta, k)$ at F.

In principle, the multiplicity l may jump as $[E]$ moves in the moduli space. However, it is natural to conjecture that l is constant along $M_{p,\beta}(\gamma)$ and takes value $l([E]) = \mu(h, d, \beta) + 1$ at all points. This is certainly in agreement with equation (4.6). While a rigorous proof is quite difficult, intuitively one expects this to be the case since the only obstructions to the normal deformations of E come from obstructions to normal deformations of the curve class β, which are independent of E.

4.2 Singular fibers and IIA/heterotic duality

In order to finish the computation one has to evaluate the contributions of the singular fibers X_σ, $\sigma \in \Sigma$ to the right hand side of equation (4.1). The presence of singularities makes a direct geometric approach difficult. However one can gain important insight using fiberwise heterotic/IIA duality for the K3 fibration $\pi : X \to \mathbb{P}^1$. Since the Donaldson-Thomas invariants are independent of the Kähler class ω, the latter can be chosen such that the volume of the section of π is much

larger than that of the K3 fibers. In this regime, it is natural to define a constructible function $\mu : \mathbb{P}^1 \to \mathbb{Z}$,

$$\mu(p) = \chi(M_p(\gamma), \nu). \tag{4.11}$$

Clearly, the value of μ at p represents the contribution of the fiber X_p to the Donaldson-Thomas invariant. More concretely, one can write

$$\Omega(\gamma) = \chi(\mathbb{P}^1, \mu). \tag{4.12}$$

In this context, fiberwise heterotic/IIA duality relates the K3 fibration $\pi : X \to \mathbb{P}^1$ to a family of world-sheet conformal field theories for the $E_8 \times E_8$ heterotic string on T^4. Therefore, given the physical interpretation of Donaldson-Thomas invariants, it is natural to expect the value of μ at a point $p \in \mathbb{P}^1$ to be related to degeneracies of perturbative BPS heterotic string states. A concrete conjecture will be formulated below.

First recall that six dimensional heterotic/IIA duality identifies a T^4 compactification of the $E_8 \times E_8$ heterotic string to a K3 compactification of the IIA string. The heterotic Narain lattice $\Gamma_{4,20}$ is isomorphic to a direct sum $U \oplus \Lambda_{K3}$, where $\Lambda_{K3} \simeq U^{\oplus 3} \oplus \Lambda_{E_8}(-1)^{\oplus 2}$ is the middle homology lattice of a smooth generic K3 surface. This identification singles out a topologically nontrivial circle $S^1_A \subset T^4$ corresponding to the first U summand.

The conformal field theory moduli space $\mathcal{M}_{\mathsf{het}}$ of the $E_8 \times E_8$ heterotic string on T^4 is a quotient of the form

$$\mathcal{M}_{\mathsf{het}} = O(\Gamma_{4,20}) \backslash \widetilde{\mathcal{M}}_{\mathsf{het}}$$

where $\widetilde{\mathcal{M}}_{\mathsf{het}} = O(4, 20)/O(4) \times O(2)$ and $O(\Gamma_{4,20})$ is the automorphism group of the Narain lattice, acting naturally on the coset space. The latter is isomorphic to the grassmannian of space-like 4-planes $\Pi \subset \Gamma_{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$, hence it is a smooth complex manifold. The quotient by the T-duality group will have orbifold singularities.

According to [3, Thm. 6], a certain open subspace of this moduli space is precisely identified with the moduli space of conformally invariant nonlinear sigma models with target space K3, including metric and B-field moduli. Hence $\mathcal{M}_{\mathsf{het}}$ is in fact a compactifiction of the sigma model moduli space. As explained for example in [3, Sect. 4.3] certain points in $\mathcal{M}_{\mathsf{het}}$ correspond to nonperturbative IIA compactification on K3 surfaces with ADE quotient singularities. Such points are associated to six dimensional gauge symmetry enhancement. More precisely, for a generic point in the moduli space, the six dimensional gauge group is $U(1)^{24}$. Let $\widetilde{\mathcal{D}} \subset \widetilde{\mathcal{M}}_{\mathsf{het}}$ be the locus where the spacelike 4-plane Π is orthogonal to some vector $\beta \in \Gamma_{4,20}$, $\beta^2 = -2$. Then the abelian gauge group is enhanced to $SU(2) \times U(1)^{23}$ at generic points on $\mathcal{D} = O(\Gamma_{4,20}) \backslash \widetilde{\mathcal{D}} \subset \mathcal{M}_{\mathsf{het}}$. For the purposes of the present discussion, it should be emphasized that the points on $\mathcal{D}$ parameterize smooth well behaved heterotic conformal field theories, although the corresponding K3 surfaces in IIA theory develop A_1 singularities.

The duality also leads to a precise indentification of the Hilbert spaces of six dimensional BPS states in the two string theories, as explained for example in [27, 12]. As shown in [12, Sect. 6.2], D4-D2-D0 BPS states with charge vector $\gamma = (r, \beta, k)$ supported on S are in one-to-one correspondence with certain perturbative heterotic string states with momentum k and winding number r on the circle S_A^1. These states are obtained by tensoring the ground state of the right moving superconformal sector with a level N states of the bosonic left moving sector, where N is determined by level matching:

$$N = r(k - r) - (r - 1)\beta^2/2 + 1. \tag{4.13}$$

These are the Dabholkar and Harvey states considered in [13] in relation to black hole entropy.

Assuming S to be algebraic, note that N is half the dimension of the moduli space of stable torsion free sheaves on S. Since the left moving sector consists of 24 bosons, it follows that the degeneracy of these states is the N-th coefficient c_N in the expansion of $q/\eta(q)^{24}$, in agreement with Goettsche's formula (4.7).

Finally, suppose S is a singular algebraic K3 surface with a single node corresponding to a generic point in $\widetilde{\mathcal{D}}$, and let $\widetilde{S}$ be its minimal crepant resolution. Let $\beta_C \subset H^2(\widetilde{S}, \mathbb{Z})$ denote the Poincaré dual of the exceptional (-2)-curve $C \subset \widetilde{S}$. Note that β_C is identified with a root vector of one of the E_8 sublattices of $H_2(\widetilde{S}, \mathbb{Z})$. As explained above, the dual heterotic conformal field theory associated to S is still smooth, except that the six dimensional gauge group of the corresponding six dimensional vacuum is enhanced to $SU(2)$. The extra massless W-bosons correspond to heterotic vertex operators associated to the root β_C. In particular the six dimensional theory exhibits a gauge symmetry which maps $\beta_C \mapsto -\beta_C$. This is the action of the generator of the Weyl group of the enhanced $SU(2)$ gauge group.

The degeneracies of DH states in the conformal theory associated to the nodal surface S are exactly the same as those computed in the conformal field theory associated to the blow-up $\widetilde{S}$. This is manifest from the counting algorithm, which is independent of deformations of conformal field theory as long as the theory remains smooth. However, since the reflection $\beta_C \mapsto -\beta_C$ is a gauge symmetry, any two states DH related by this reflection are physically identical, so such a pair should be counted only once in the six dimensional BPS spectrum.

Returning to the family of Λ-polarized algebraic K3 surfaces $\pi : X \to \mathbb{P}^1$, note that this family cannot be canonically identified with a family of heterotic conformal field theories since the restriction of the Calabi-Yau threefold metric to a K3 fiber need not be hyper-Kähler. However, the BPS index for D4-D2-D0 states supported on a fiber is independent of the metric, hence one can employ similar arguments in this context as well.

Using equations (4.5), (4.6) and (4.8), the contribution of the fiber X_p, $p \in U^\pi$, to the four

dimensional BPS index of charge $\gamma = (r, d, n)$ is given by

$$\mu(p) = -c(r(n-r) + d^2/2 + 1)\delta_{d,\Lambda} + \sum_{\substack{h \in \mathbb{Z}, \\ r(r-n) \leq h \leq d^2/2+1}} \sum_{\beta \in \mathcal{B}^{\mathrm{iso}}(h,d)} \mu(h, d, \beta)c(r(n-r) + h). \quad (4.14)$$

The sum in the right hand side represents the contribution of all charge vectors $\beta \in H^2(X_p, \mathbb{Z})$ which yield the same charge vector α with respect to the four-dimensional abelian gauge group i.e. $\iota_*\beta = \alpha$. Aside from the weights $\mu(h, \alpha, \beta)$, the contribution of each class β is given by the corresponding degeneracy of heterotic DH states. As explained below (4.10), for any class $\beta \in \mathcal{B}_p^{\mathrm{iso}}(h, d)$, the weight $\mu(h, d, \beta)$ represents the vacuum multiplicity of the corresponding BPS D4-D2-D0 configurations supported on X_p.

Moreover, recall that the components $M_{p,\alpha}(\gamma)$ of the moduli space of D4-D2-D0 branes on X_p fit in the smooth family $M_U^\infty(\gamma)$ over U. Therefore, according to formula (2.15), the contribution of degeneracies of states with $\beta = \alpha$ to the four dimensional index should be weighted by (-1). This sign encodes their four dimensional spin quantum number.

By analogy with six dimensional heterotic/IIA duality, it follows that the contribution $\mu(\sigma)$, $\sigma \in S^\pi$, of a singular fiber can be inferred from the contribution (4.14) of its blow-up $\widetilde{X}_\sigma$. The main point is that, choosing appropriate hyper-Kähler metrics on $X_\sigma, \widetilde{X}_\sigma$, one obtains smooth heterotic conformal field theories related by a smooth deformation. Hence the DH degeneracies as well as the four dimensional spin quantum number will be the same in the two theories. The only difference is the gauge symmetry $\beta_C \mapsto -\beta_C$ in the six dimensional vacuum associated to the nodal surface, which implies that DH states with charge $\alpha + l\beta_C$ are physically indistinguishable from DH states with charge $\alpha - l\beta_C$. Working under the genericity assumptions formulated in Section 3, the multiplicity of all curve classes supported on $\widetilde{X}_\sigma$ is given by equation (3.11). Therefore, collecting the facts, one is led to the following conjectural expression

$$\mu(\sigma) = -c(r(n-r) + d^2/2 + 1)\delta_{d,\Lambda} + \sum_{\substack{l \in \mathbb{Z}, \, l \geq 1 \\ l^2 \leq r(n-r)+d^2/2+1}} c(r(n-r) + d^2/2 + 1 - l^2)\delta_{d,\Lambda} \quad (4.15)$$

for the contribution of a singular nodal fiber to the BPS index. Using equations (4.14) and (4.15) in equation (4.12), one then obtains

$$\Omega(\gamma) = -\chi(\mathbb{P}^1)c(r(n-r) + d^2/2 + 1)\delta_{d,\Lambda} \sum_{\substack{h \in \mathbb{Z} \\ r(r-n) \leq h < d^2/2+1}} \sum_{\beta \in \mathcal{B}^{\mathrm{iso}}(h,d)} \mu(h, d, \beta)c(r(n-r) + h)$$

$$+ |S^\pi| \sum_{\substack{l \in \mathbb{Z}, \, l \geq 1 \\ l^2 \leq r(n-r)+d^2/2+1}} c(r(n-r) + d^2/2 + 1 - l^2)\delta_{d,\Lambda}.$$

Moreover, using equations (3.9), (3.13) the above formula can be rewritten as

$$\Omega(\gamma) = \frac{1}{2} \sum_{\substack{h \in \mathbb{Z} \\ r(r-n) \leq h \leq d^2/2+1}} c(r(n-r)+h)\widetilde{NL}_{h,d} \tag{4.16}$$

where $\widetilde{NL}_{h,d}$ are the Noether-Lefschetz numbers of the threefold $\widetilde{X}$ constructed in Section 3.

4.3 Generating functions for primitive charge vectors

Suppose the pair $(r,d) \in \mathbb{Z}_{\geq 1} \times \Lambda^\vee$ is primitive. Then any charge vector $\gamma = (r,d,n)$ will be primitive. Recall that the rank r partition function of vertical D4-D2-D0 invariants is an expression of the form (2.9), where

$$Z_{BPS}(X,r,\delta;\tau) = \sum_{n \in \mathbb{Z}} \Omega(r,d,n)q^{n+d^2/2r-r}.$$

In the above formula $\delta = [d]_r \in \Lambda^\vee/r\Lambda$ is the equivalence class of $d \in \Lambda^\vee$. Let $h(d) \in \mathbb{Z}$ be defined by

$$\frac{d^2}{2} + 1 = \frac{h(d)}{2m}$$

where $m = \det(M)$. Using equation (4.16),

$$Z_{BPS}(X,r,\delta;\tau) = \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ rn \geq r^2-h(d)/2m}} q^{n+d^2/2r-r} \sum_{r(r-n) \leq h \leq h(d)/2m} c_{r(n-r)+h}\widetilde{\Phi}_{[d]_1}[h(d)/2m - h],$$

where $[d]_1 \in \Lambda^\vee/\Lambda$ denotes the equivalence class of $d \bmod \Lambda$. Let

$$\frac{l}{2m} = r(n-r) + \frac{h(d)}{2m}, \qquad k = r(n-r) + h$$

with $k,l \in \mathbb{Z}$. Therefore

$$n = r + \frac{l - h(d)}{2rm}.$$

Then the right hand side of the above equation can be written as

$$\frac{1}{2}q^{d^2/2r} \sum_{\substack{l \in \mathbb{Z}, \, l \geq 0, \\ (l-h(d))/2rm \in \mathbb{Z}}} q^{(l-h(d))/2rm} \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq l/2m}} c_k \widetilde{\Phi}_{\bar{d}}[l/2m - k] =$$

$$\frac{1}{4rl}q^{d^2/2r} \sum_{l \in \mathbb{Z}, \, l \geq 0} q^{(l-h(d))/2rm} \sum_{s=0}^{2rm-1} e^{2\pi i(l-h(d))s/2rm} \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq l/2m}} c_k \widetilde{\Phi}_{\bar{d}}[l/2m - k].$$

Let $\Delta(q) = \eta(q)^{24}$ and note that $\Delta^{-1}(q) = q^{-1}\sum_{k \geq 0} c(k)q^k$ by Goettsche's formula (4.7). Then

$$\sum_{l \in \mathbb{Z}, \, l \geq 0} q^{l/2rm}e^{2\pi ils/2rm} \sum_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq l/2m}} c_k \widetilde{\Phi}_\delta[l/2m - k] = q^{1/r}e^{2\pi is/r}\left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau + s}{r}\right).$$

Therefore

$$
\begin{aligned}
Z_{BPS}(X, r, \delta; \tau) &= \frac{1}{4rm} q^{d^2/2r} q^{(1-h(d)/2m)/r} \sum_{s=0}^{2rm-1} e^{2\pi i s(1-h(d)/2m)/r} \left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau+s}{r}\right) \\
&= \frac{1}{4rm} \sum_{s=0}^{2rm-1} e^{-\pi i s d^2/r} \left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau+s}{r}\right).
\end{aligned}
\tag{4.17}
$$

As discussed in detail in Section 3.3, the collection $\left(\widetilde{\Phi}_{[d]_1}\right)$ form a vector valued modular form of weight $(22-\ell)/2$ and type ρ_Λ^*. Hence the collection $\left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)$ also form a vector valued modular form of the same type and weight $(-1-\ell/2)$. In particular,

$$
\left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau+s+kr}{r}\right) = e^{\pi i k d^2} \left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau+s}{r}\right).
$$

At the same time

$$
e^{-\pi i(s+kr)d^2/r} = e^{-\pi i k d^2} e^{-\pi i s d^2/r}.
$$

Therefore each term in the right hand side of equation (4.17) is invariant under $s \mapsto s+kr$, $k \in \mathbb{Z}$. Then equation (4.17) can be simplified to

$$
Z_{BPS}(X, r, \delta; \tau) = \frac{1}{2r} \sum_{s=0}^{r-1} e^{-\pi i s d^2/r} \left(\Delta^{-1}\widetilde{\Phi}_{[d]_1}\right)\left(\frac{\tau+s}{r}\right).
\tag{4.18}
$$

In particular, for $r = 1$ one obtains $Z_{BPS}(X, 1, \delta; \tau) = \left(\Delta^{-1}\Phi_\delta^{\tilde{\pi}}\right)(\tau)$ for any $\delta \in \Lambda^\vee/\Lambda$, in agreement with the results of [20]. As required by physical arguments [14, 18], the collection $(Z_{BPS}(X, 1, g; \tau))_{g \in \Lambda^\vee/\Lambda}$ is a weight $-1 - \ell/2$ vector-valued modular form with values in the Weil representation.

4.4 Non-primitive charge vectors and multicover contributions

To conclude this section suppose γ is not primitive. Then in general there will exist strictly semistable objects in the moduli space of stable sheaves, making the theory of Donaldson-Thomas invariants more difficult. In particular one has the rational invariants $DT(\gamma)$ which are related to the integral ones $\Omega(\gamma)$ by the multicover formula (2.16),

$$
DT(r, d, n) = \sum_{\substack{k \in \mathbb{Z}, k \geq 1 \\ (r,d,n)=k(r',d',n')}} \frac{1}{k^2} \Omega(r', d', n').
$$

For any pair (r, δ), $\delta \in \Lambda^\vee/r\Lambda$, let

$$
Z_{DT}(X, r, \delta; \tau) = \sum_{n \in \mathbb{Z}} DT(r, d, n) q^{n+d^2/2r-r}.
$$

Note that the right hand side of the above equation depends only on the equivalence class δ of d mod $r\Lambda$ since the rational Donaldson-Thomas invariants are invariant under monodromy transformations (2.3) as well. In order to evaluate this series, first recall that for any pair (k, r') with $kr' = r$ there is an injective morphism

$$f_{r',k} : \Lambda^\vee / r'\Lambda \to \Lambda^\vee / r\Lambda, \qquad f_{r',k}([d]_{r'}) = [kd]_r, \quad \text{for all } d \in \Lambda^\vee.$$

Then, using the above multicover formula, the generating functional is written as

$$Z_{DT}(X, r, \delta; \tau) = \sum_{\substack{k \in \mathbb{Z}, k \geq 1 \\ (r,d) = k(r',d')}} \frac{1}{k^2} \sum_{n' \in \mathbb{Z}} q^{k(n' + d'^2/2r' - r')} \Omega(r', d', n')$$

$$= \sum_{\substack{k \in \mathbb{Z}, k \geq 1 \\ r = kr', \ \delta = f_{r',k}(\delta')}} \frac{1}{k^2} Z_{BPS}(X, r', \delta'; k\tau).$$

In the right hand side, $\delta' \in \Lambda^\vee / r'\Lambda$ is uniquely determined by (k, δ) since $f_{r,k}$ is injective.

At the next stage, one needs a generalization of the conjectural formula (4.17) to all pairs (r, d), not just primitive ones. On physics grounds, the natural conjecture at this point is that (4.17) is in fact valid for all such pairs, including non-primitive ones. The main physical argument for this conjecture is based on modularity constraints. Physics arguments based on S-duality [14] or M5-brane elliptic genus [18] imply that the collection of partition functions $Z_{DT}(X, r, \delta; \tau)$, $\delta \in \Lambda^\vee / r\Lambda$ must be a weight $-1 - \ell/2$ vector valued modular form for the Weil representation of $\sqrt{r}\Lambda$. This is in fact proven in Section 6 for the explicit series in equation (4.19). Although (r, δ) is non-primitive, for any representative $d \in \Lambda^\vee$ of δ, there are infinitely many values of $n \in \mathbb{Z}$ such that $\gamma = (r, d, n)$ is primitive. By the arguments of the previous section, the conjectural formula (4.16) will apply to all such values. Then the generating function $Z_{DT}(X, r, \delta; \tau)$ should be given by the right hand side of equation (4.17) since the vector space of weight $-1 - \ell/2$ vector valued modular forms for a given Weil representation is finite dimensional as explained in section 6. Then one obtains

$$Z_{DT}(r, \delta; \tau) = \sum_{\substack{k \in \mathbb{Z}, \ k \geq 1 \\ r = kr', \ \delta = f_{r',k}(\delta')}} \frac{1}{k^2} \frac{1}{2r'} \sum_{s=0}^{r'-1} e^{-\pi i s (d')^2 / r'} \left(\Delta^{-1} \Phi_{[d']_1}^{\tilde{\pi}} \right) \left(\frac{k\tau + s}{r'} \right).$$

At this point note that the $\mathbb{Q}$-valued symmetric bilinear pairing of $\Lambda^\vee$ induces a $\mathbb{Q}/\mathbb{Z}$-valued symmetric bilinear pairing on $\Lambda^\vee / r\Lambda$ given by

$$([d]_r, [d']_r) \mapsto \frac{d \cdot d'}{r}.$$

Then the formula can be further rewritten as

$$Z_{DT}(X, r, \delta; \tau) = \frac{1}{2r^2} \sum_{\substack{k \in \mathbb{Z}, \ k \geq 1 \\ r = kl, \ \delta = \tilde{f}_{l,k}(\eta)}} \sum_{s=0}^{l-1} l e^{-\pi i s l \delta^2 / r^2} \left(\Delta^{-1} \Phi_{[\eta]_1}^{\tilde{\pi}} \right) \left(\frac{k\tau + s}{l} \right), \qquad (4.19)$$

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where $[\eta]_1 \in \Lambda^\vee/\Lambda$ is the equivalence class of $\eta \in \Lambda^\vee/l\Lambda$ mod Λ. This is precisely equation (1.3).

Finally, the total rank generating function for Donaldson-Thomas invariants for fixed $r \geq 1$ is obtained by summing over $d \in \Lambda^\vee$. As shown in Section 2, this yields a sum of the form

$$Z_{DT}(X,r;\tau) = \sum_{\delta \in \Lambda^\vee/\Lambda} Z_{DT}(X,r,\delta;\tau)\Theta^*_{r,\delta}(\tau,\bar\tau),$$

where

$$\Theta^*_{r,\delta}(\tau,\bar\tau) = \sum_{\alpha \in \Lambda} e^{-2\pi i \tau(d+r\alpha)^2_-/2r - 2\pi i \bar\tau(d+r\alpha)^2_+/2r}$$

for $d \in \Lambda^\vee$ an arbitrary representative of δ. Using the multicover formula for Donaldson-Thomas one obtains

$$Z_{DT}(X,r;\tau) = \sum_{\delta \in \Lambda^\vee/r\Lambda} \sum_{\substack{k \in \mathbb{Z},\ k \geq 1 \\ r=kr',\ \delta=f_{r',k}(\delta')}} \frac{1}{k^2} Z_{BPS}(X,r',\delta';k\tau)\Theta^*_{r,\delta}(\tau,\bar\tau).$$

Moreover a simple computation shows that

$$\Theta^*_{r,\delta}(\tau,\bar\tau) = \Theta^*_{r',\delta'}(k\tau,k\bar\tau).$$

Then, using equation (4.17), one then obtains by straightforward computations

$$Z_{DT}(X,r;\tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z},\ k,l \geq 1 \\ kl=r}} \sum_{\eta \in \Lambda^\vee/l\Lambda} l\big(\Delta^{-1}\widetilde{\Phi}_{[\eta]}\big)\left(\frac{k\tau+s}{l}\right)\Theta^*_{l,\eta}\left(\frac{k\tau+s}{l},\frac{k\bar\tau+s}{l}\right), \qquad (4.20)$$

which is precisely equation (1.4).

As noted in Section 1.2, the above formula reduces to a Hecke transform if the intersection form on Λ is unimodular. If that is the case, $\Lambda^\vee/\Lambda$ is trivial and the Noether-Lefschetz numbers are encoded in an ordinary modular form $\widetilde{\Phi}$. Then in the right hand of (4.20) one obtains a sum of the form

$$\sum_{\eta \in \Lambda/l\Lambda} \Theta^*_{l,\eta}\left(\frac{k\tau+s}{l},\frac{k\bar\tau+s}{l}\right).$$

This sum can be easily shown to be equal to $\Theta^*\left((k\tau+s)/l,(k\bar\tau+s)/l\right)$, there $\Theta^*(\tau,\bar\tau)$ is the complex conjugate of the non-holomorphic Siegel theta function associated to the self-dual lattice. Then equation (4.20) specializes indeed to (1.5).

5 Recursive derivation from stable pair invariants

The goal of this section is to provide an alternative derivation for the main formula (1.3) based on the mathematical results of [22]. Using wallcrossing techniques, vertical Donaldson-Thomas invariants

for nodal K3 pencils are related in loc. cit. to the stable pair invariants defined by Pandharipande and Thomas in [50]. It will be shown in this section that this relation determines the Donaldson-Thomas invariants in terms of stable pair ones by a recursive algorithm. Moreover, assuming Gromv-Witten/stable pair correspondence and the multicover formula (2.16), this algorithm yields a proof of the main formula (1.3). Using the results of Maulik and Pandharipande on Gopakumar-Vafa invariants of K3 pencils, the recursive algorithm reduces the proof of the main formula to a remarkable combinatorial identity which is proven in Section 5.4.

5.1 Background

Let X be a nonsingular projective threefold, $v \in H_2(X, \mathbb{Z})$ a curve class, and $n \in \mathbb{Z}$. According to [50], the moduli space of stable pairs $\mathrm{PT}_n(X, v)$ parametrizes the pairs $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ where $\mathcal{F}$ is a pure 1-dimensional sheaf on X with $\mathrm{ch}_2(\mathcal{F}) = v$ and $\chi(\mathcal{F}) = n$ such that the cokernel of s is 0-dimensional. It was shown by the authors that $\mathrm{PT}_n(X, v)$ is a locally complete moduli space of complexes in the derived category, which enabled them to construct a perfect obstruction theory on $\mathrm{PT}_n(X, v)$. In the case where the virtual dimension of the moduli space is zero, the stable pair invariants $P_{n,v}$ are defined by taking the degree of the virtual cycle obtained from this obstruction theory.

Now suppose X has a $K3$ fibration structure $\pi : X \to C$ over a smooth projective curve C with at most nodal fibers. The sublattice of vertical classes will be denoted by $H_2(X, \mathbb{Z})^\pi = \mathrm{Ker}(\pi_*) \subset H_2(X, \mathbb{Z})$. Since the canonical class of X is vertical, it is easy to show that the virtual dimension of the moduli space of stable pairs is zero for any $v \in H_2(X, \mathbb{Z})^\pi$ and any $n \in \mathbb{Z}$. Such stable pairs, will be called *"vertical stable pairs"*. The vertical stable pair series of X is then defined by

$$PT^{\mathsf{vert}}(X) = \sum_{\substack{n \in \mathbb{Z}, \\ v \in H_2(X, \mathbb{Z})^\pi}} P_{n,v} \, q^n t^v.$$

Note that in the above formula one has to sum only over nonzero, effective curve classes. This condition will be written as $v > 0$.

For X a smooth Calabi-Yau threefold, of the main results of [22] relates the vertical stable invariants of X to the vertical DT invariants $DT(r, v, n)$ introduced in Section 2.3. Recall that the latter are virtual counting invariants for semistable sheaves E on X with numerical invariants

$$\mathrm{ch}_0(E) = 0, \qquad \mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = v, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x),$$

where $x \in X$ is an arbitrary closed point. Therefore they are related to the invariants used in [22, Thm. 2] by $DT(r, v, n) = J(r, v, -n)$. Assuming a certain technical conjecture, [22, Conjecture

2.3], the authors of [22] proved the following identity [22, Thm. 2] using wall-crossing techniques:

$$\mathrm{PT}^{\mathsf{vert}}(X) = \prod_{r \geq 0, v > 0, n \leq 0} \exp\left((-1)^{n-1} DT(r, v, n) q^{-n} t^\beta\right)^{-n+2r}$$

$$\cdot \prod_{r > 0, v > 0, n < 0} \exp\left((-1)^{n-1} DT(r, \beta, n) q^n t^\beta\right)^{-n+2r}.$$

$$(5.1)$$

In order to make contact with the set up of Sections 3 and 2.3 suppose the K3 fibration $\pi :$ $X \to C$ satisfies conditions $(a) - (d)$ in Section 3. Then Poincaré duality yields an isomorphism $H_2(X, \mathbb{Z})^\pi \simeq \Lambda^\vee$, where Λ is the polarizing lattice for the pencil $\widetilde{X}$. Therefore the vertical classes v will be identified with dual lattice vectors $d \in \Lambda^\vee$. The next subsection will provide a recursive method for computing the invariants $DT(r, d, n)$ on the right hand side of (5.1) directly from the partition function $\mathrm{PT}(X)$ of $P_{n,d}$ invariants on the left hand side.

5.2 A recursive algorithm for $DT(r, d, n)$

The recursive algorithm for the invariants $\{DT(r, d, n)\}$ is based on the following two key properties.

1. $DT(r, d, n) \neq 0$ implies either $r = 0$, or

$$d^2 + 2rn \geq 0. \tag{5.2}$$

2. For any (r, d, n) there is an identity

$$DT(r, d, n) = DT(r, d + r\alpha, n - r\alpha^2/2 - rd \cdot \alpha).$$

The first property follows from Bogomolov inequality. In more detail, the proof proceeds in several steps, as follows. Suppose first E is a stable sheaf on X supported on a smooth $K3$ fiber $\iota_p : X_p \hookrightarrow X$, with invariants (r, d, n). Then E is the extension by zero of a stable sheaf F on X_p with invariants

$$\mathrm{rk}(F) = r, \qquad c_1(F) = \beta, \qquad c_2(F) = k$$

such that

$$\iota_{p*}(\beta) = d, \qquad k - \frac{\beta^2}{2} = n.$$

If $r > 0$, F must be torsion free and Bogomolov inequality reads

$$k - \frac{r-1}{2r} \beta^2 \geq 0,$$

which yields

$$\beta^2 + 2rn \geq 0.$$

As explained in Section 3.3, the algebraic Hodge theorem yields a second inequality,

$$\beta^2 \leq d^2.$$

This proves inequality (5.2). Next, suppose E is a stable sheaf scheme theoretically supported on a nodal fiber $\iota_\sigma : X_\sigma \to X$. In that case inequality (5.2) is proven by analogy with Lemmas 4.3 and 4.4 in [15]. Finally, suppose E is semistable. Then inequality (5.2) is determined by from the above results for stable shaves using a Jordan-Hölder filtration by analogy with Lemma 4.5 in [15].

As noted in Section 2.3, the second property follows from the observation that the tensor product by the line bundle $\mathcal{O}_X(H)$ yields an isomorphism of moduli stacks of semistable sheaves for any $H \in \Lambda \subset \mathrm{Pic}(X)$.

In order to construct the recursive algorithm for $DT(r, d, n)$, note that equation (5.1) yields

$$\begin{aligned}
\log(\mathrm{PT}(X)) &= \sum_{r \geq 0, d > 0, n \leq 0} (-1)^{n-1}(-n + 2r)DT(r, d, n)q^{-n}t^d \\
&+ \sum_{r > 0, d > 0, n < 0} (-1)^{n-1}(-n + 2r)DT(r, d, n)q^n t^d.
\end{aligned} \tag{5.3}$$

Then one proceeds inductively by $r \in \mathbb{Z}$, $r \geq 0$.

Step 1. $r = 0$. According to property (2) above, $DT(0, d, n) = DT(0, d + \alpha, n - d \cdot \alpha)$ for any divisor $H \in \Lambda \subset \mathrm{Pic}(X)$. Now note that if H is sufficiently relatively ample with respect to the projection map $\pi : X \to \mathbb{P}^1$, the coefficient of $q^{-n+d\cdot\alpha}t^{d+\alpha}$ on the right hand side of (5.3) does not contain any $r > 0$ terms. Indeed if otherwise it does, then inequality (5.2) implies

$$d^2 + 2r(n - d \cdot \alpha) \geq 0.$$

Since $d > 0$, this leads to a contradiction for sufficiently ample H, keeping (d, n) fixed. Therefore the invariant $DT(0, d, n)$ is determined by the coefficient of $q^{-n+d\cdot h}t^d$ for $H \gg 0$ in the left hand side of (5.3).

Step 2. Induction on $r > 0$ and computation of $DT(r, d, n)$ invariants: Now apply the induction on r. Suppose that $r > 0$ and every $N(r', \beta, n)$ is expressed with respect to the stable pair invariants for $r' < r$. By property (2) above,

$$DT(r, d, n) = DT(r, d + r\alpha, n - r\alpha^2/2 - d \cdot \alpha)$$

for any $H \in \Lambda$. Let $\gamma = (r, d, n)$ and $N(\gamma, \alpha) = -n + r\alpha^2/2 + d \cdot \alpha$. Again, the key observation is that if H is sufficiently relatively ample, the coefficient of $q^{N(\gamma,\alpha)}t^{d+r\alpha}$ in the right hand side of

(5.3) does not contain any invariants

$$DT(R, d+h, -N(\gamma, \alpha))$$

with $R > r$. Indeed if it does otherwise, then property (2) implies that

$$(d + r\alpha)^2 + 2R(n - r\alpha^2/2 - d \cdot \alpha) \geq 0$$

This leads to a contradiction for any $R > r$ if H is sufficiently relatively ample, keeping (r, d, n) fixed. $h \gg 0$. Therefore for such divisors H, the sum (5.3)

$$(-1)^{N(\gamma, \alpha)-1} \sum_{s=0}^{r} (N(\gamma, \alpha) + 2s) DT(s, d + r\alpha, -N(\gamma, \alpha))$$

is identified with the coefficient of $q^{N(\gamma, \alpha)} t^{d+r\alpha}$ in the left hand side of (5.3). Using the induction, one can then express $DT(r, d, n)$ in terms of stable pair invariants.

5.3 Vertical Donaldson-Thomas invariants from Gopakumar-Vafa invariants

The goal of this section is to show that the the conjectural formula (4.16) follows from the above recursive algorithm assuming the Gromov-Witten/stable pair correspondence to hold in the current setup. If this is the case, the vertical stable pair partition function can be written in terms of vertical Gopakumar-Vafa invariants, which have been computed in [41]:

$$\log(\mathrm{PT}(X)) = \sum_{g \geq 0} \sum_{\substack{d \in \Lambda^\vee \\ d > 0}} \sum_{k=1}^{\infty} \frac{(-1)^{g-1}}{k} n_{g,d}(X) \left(\frac{(1-(-q)^k)^2}{(-q)^k} \right)^{g-1} t^{kd} \tag{5.4}$$

where $n_{g,d}(X)$ are the vertical Gopakumar-Vafa invariants of X. According to [41, Thm. 1],

$$n_{g,d}(X) = \frac{1}{2} \sum_{h=g}^{\infty} r_{g,h} \widetilde{NL}_{h,d} \tag{5.5}$$

where $r_{g,h}$ are the local Gopakumar-Vafa invariants of a smooth K3 surface and $\widetilde{NL}_{h,d}$ are the Noether-Lefschetz numbers of the K3 pencil $\tilde{\pi} : \widetilde{X} \to \Sigma$ constructed in Section 3. An explicit conjecture for the invariants $r_{g,h}$ was formulated by Katz, Klemm and Vafa [31], and proven by Pandharipande and Thomas [49]. This reads

$$\sum_{g \geq 0} \sum_{h \geq 0} (-1)^g r_{g,h} \left(\frac{(1-y)^2}{y} \right)^{2g} q^h = \prod_{n \geq 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}. \tag{5.6}$$

The above formula implies in particular that $r_{g,h} = 0$ for $g > h$.

According the inductive step in the previous section, for a given triple $\gamma = (r, d, n)$, one has to choose a sufficiently relatively ample divisor $H = \alpha \in \Lambda \subset \mathrm{Pic}(X)$. Then the linear combination

$$(-1)^{N(\gamma,\alpha)-1} \sum_{s=0}^{r} (N(\gamma,\alpha) + 2s) DT(s, d + r\alpha, -N(\gamma,\alpha)) \tag{5.7}$$

is identified with the the coefficient $L_{N,d+r\alpha}$ of $q^{N(\gamma,\alpha)} t^{d+r\alpha}$ in (5.4), where $\alpha = DH$ and $N(\gamma,\alpha) = -n + r\alpha^2/2 + d \cdot \alpha$. Note that $L_{N,d+r\alpha}$ can be written as

$$L_{N,d+r\alpha} = L^0_{N,d+r\alpha} + L^{\geq 2}_{N,d+r\alpha}$$

where $L^0_{N,d+r\alpha}$ is the contribution of genus $g = 0$ terms while $L^{\geq 2}_{N,d+r\alpha}$ encodes the terms with $g \geq 2$. Expanding the right hand side of (5.4) in powers of q yields

$$L^0_{N,d+r\alpha} = (-1)^{N(\gamma,h)-1} N(\gamma,h) \sum_{\substack{k \in \mathbb{Z},\ k \geq 1 \\ k|(d+r\alpha, N(\gamma,h))}} \frac{1}{k^2} n_{0,(d+r\alpha)/k} \tag{5.8}$$

and

$$L^{\geq 2}_{N,d+r\alpha} = (-1)^N \sum_{\substack{k \in \mathbb{Z},\ k \geq 1 \\ k|(d+r\alpha, N(\gamma,h))}} \sum_{g \geq 1+N/k} \frac{(-1)^{N/k}}{k} n_{g,(d+r\alpha)/k} \binom{2g-2}{g-1-N/k}. \tag{5.9}$$

Note that for $r = 0$, the expression (5.7) reduces to

$$(-1)^{N(\gamma,h)-1} N(\gamma,h) DT(0, d + r\alpha, -N(\gamma,h))$$

Using the multicover formula for Donaldson-Thomas invariants, this is further equal to

$$(-1)^{N(\gamma,h)-1} N(\gamma,h) \sum_{\substack{k \in \mathbb{Z},\ k \geq 1 \\ k|(d+r\alpha, N(\gamma,h))}} \frac{1}{k^2} \Omega(0, (d + r\alpha)/k, -N(\gamma,h)/k).$$

Moreover, as conjectured in [30], for any $v \in H^2(X, \mathbb{Z})$ and any $N \in \mathbb{Z}$, one has $\Omega(0, v, N) = n_{0,v}$. Therefore $L^0_{N,d+r\alpha}$ in equation (5.8) equals the $s = 0$ term in (5.7).

Next, using (5.5), the higher genus contributions $L^{\geq 2}_{N,d+r\alpha}$ can be written in terms of the local invariants $r_{g,h}$ and Noether-Lefschetz numbers. Note that $N^{\tilde{\pi}}_{h,d} = 0$ for $h > d^2/2 + 1$ and $r_{g,h} = 0$ for $g > h$. This implies that

$$\frac{N(\gamma,\alpha)}{k} + 1 \leq g \leq h \leq \frac{(d+r\alpha)^2}{2k^2} + 1 \tag{5.10}$$

for any $r_{g,h}$ occuring by substitution in the right hand side of equation (5.9). Then equations (5.5), (5.9), yield

$$L^{\geq 2}_{N,d+r\alpha} =$$

$$\sum_{\substack{k \in \mathbb{Z},\ k \geq 1 \\ k|(d+r\alpha, N(\gamma,h))}} \frac{(-1)^{(k+1)N(\gamma,\alpha)/k}}{2k} \sum_{\substack{h \in \mathbb{Z} \\ N(\gamma,\alpha)/k+1 \leq h \leq (d+r\alpha)^2/2k^2+1}} N^{\tilde{\pi}}_{h,(d+r\alpha)/k} \sum_{g=1+N/k}^{h} r_{g,h} \binom{2g-2}{g-1-N/k}. \tag{5.11}$$

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At the same time, using the multicover formula for Donaldson-Thomas invariants, the $s \geq 1$ part of the sum in (5.7) becomes

$$(-1)^{N(\gamma,\alpha)-1} \sum_{s=0}^{r} (N(\gamma,\alpha) + 2s) \sum_{\substack{k \in \mathbb{Z}, \ k \geq 1 \\ k|(s,d+r\alpha,N(\gamma,\alpha))}} \frac{1}{k^2} \Omega(s/k, (d+r\alpha)/k, -N(\gamma,\alpha)/k) =$$

$$(-1)^{N(\gamma,\alpha)-1} \sum_{\substack{k \in \mathbb{Z}, \ k \geq 1 \\ k|(d+r\alpha,N(\gamma,h))}} \sum_{\substack{u \in \mathbb{Z} \\ 1 \leq u \leq r/k}} \frac{1}{k} \left(\frac{N(\gamma,\alpha)}{k} + 2u \right) \Omega(u, (d+r\alpha)/k, -N(\gamma,\alpha)/k).$$

(5.12)

Now recall the conjectural formula (4.16) for the vertical D4-D2-D0 degeneracies. For any triple $\gamma = (r, d, n)$, $r \geq 1$, one has

$$\Omega(\gamma) = \frac{1}{2} \sum_{\substack{h \in \mathbb{Z} \\ r(r-n) \leq h \leq d^2/2+1}} c(r(n-r) + h) \widetilde{NL}_{h,d}$$

where

$$\sum_{n \geq 0} c(n) q^n = \frac{1}{\prod_{n \geq 1}(1 - q^n)^{24}}.$$

By comparison with equation (5.5), it follows that $c(n) = r_{0,n}$ for any $n \geq 0$. Then one obtains

$$\Omega(u, (d+r\alpha)/k, -N(\gamma,\alpha)/k) = \frac{1}{2} \sum_{\substack{h \in \mathbb{Z} \\ u(u+N(\gamma,\alpha)/k) \leq h \leq (d+r\alpha)^2/2k^2+1}} c(h - u(u + N(\gamma,\alpha)/k)) \widetilde{NL}_{h,(d+r\alpha)/k}$$

(5.13)

Now note that inequality

$$u \left(u + \frac{N(r,\alpha)}{k} \right) \leq \frac{(d+r\alpha)^2}{2k^2} + 1$$

implies

$$u \leq -\frac{N(\gamma,\alpha)}{2k} + \frac{1}{2k} \left(N(\gamma,\alpha)^2 + 2(d+r\alpha)^2 + 4 \right)^{1/2}.$$

For sufficiently large α, keeping $\gamma = (r, d, n)$ fixed this inequality is equivalent to $u \leq r/k$, which is precisely the upper bound on u in the sum (5.12). At the same time, since $u \geq 1$, the range of h in the right hand side of equation (5.13) is

$$\frac{N(\gamma,\alpha)}{k} + 1 \leq h \leq \frac{(d+r\alpha)^2}{2k^2} + 1.$$

Therefore expression (5.12) can be further written as

$$(-1)^{N(\gamma,\alpha)-1} \sum_{\substack{k \in \mathbb{Z}, \ k \geq 1 \\ k|(d+r\alpha,N(\gamma,h))}} \frac{1}{2k} \sum_{\substack{h \in \mathbb{Z} \\ N(\gamma,\alpha)/k+1 \leq h \leq (d+r\alpha)^2/2k^2+1}} N^{\tilde{\pi}}_{h,(d+r\alpha)/k}$$

$$\sum_{\substack{u \in \mathbb{Z}, \ u \geq 1 \\ u(u+N(\gamma,\alpha)/k) \leq h}} \left(\frac{N(\gamma,\alpha)}{k} + 2u \right) r_{0,h-u(u+N(\gamma,\alpha)/k)}.$$

(5.14)

In order to facilitate comparison with (5.11), note that the latter reads

$$L_{N,d+r\alpha}^{\geq 2} =$$

$$\sum_{\substack{k\in\mathbb{Z},\ k\geq 1 \\ k|(d+r\alpha,N(\gamma,h))}} \frac{(-1)^{(k+1)N(\gamma,\alpha)/k}}{2k} \sum_{\substack{h\in\mathbb{Z} \\ N(\gamma,\alpha)/k+1\leq h\leq (d+r\alpha)^2/2k^2+1}} N_{h,(d+r\alpha)/k}^{\tilde{\pi}} \sum_{g=1+N/k}^{h} r_{g,h}\binom{2g-2}{g-1-N/k}.$$

Then one learns that the two expressions are in agreement provided that the following identity holds for any $h, n \in \mathbb{Z}$, $n \geq 1$, $h \geq n+1$:

$$\sum_{g=n+1}^{h} r_{g,h}\binom{2g-2}{g-n-1} = (-1)^{n-1} \sum_{\substack{s\in\mathbb{Z},\ s\geq 1 \\ s(s+n)\leq h}} (n+2s)r_{0,h-s(s+n)}. \tag{5.15}$$

This is rigorously proven in the next subsection.

5.4 The combinatorial identity

Identity (5.15) is equivalent to

$$\sum_{s=1}^{\infty}(n+2s)r_{0,h-s(s+n)} = (-1)^{n-1}\sum_{g=n+1}^{h}\binom{2g-2}{g-n-1}r_{g,h}. \tag{5.16}$$

for any $n \in \mathbb{Z}$, $0 \leq n \leq |h|$, provided that, by convention, $r_{0,k} = 0$ for $k < 0$.

In order to prove this, first recall the relevant statements of [41]. The $r_{g,h}$ are given by

$$\sum_{g\geq 0}\sum_{h\geq 0}(-1)^g r_{g,h}\left(y^{\frac{1}{2}}-y^{-\frac{1}{2}}\right)^{2g}q^h = \prod_{n\geq 0}\frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}$$

where the right-hand side is a meromorphic function on $\mathbb{H}\times\mathbb{C}$ and this identity only holds if $|qy| < 1$ and $|y^{-1}q| < 1$. Suppose in addition $|y| < 1$. Then this formula has two consequences, first $r_{g,h} = 0$ if $g > h$ and also in the specialization $y = 1$ it reduces to

$$\sum_{h\geq 0}r_{0,h}q^h = \prod_{n\geq 1}\frac{1}{(1-q^n)^{24}}.$$

Secondly we recall the meromorphic index minus two Jacobi form $\Phi_2(y,q)$ of [8], actually its rescaled version, rescaled by $\eta(q)^{-18}$ is

$$\frac{\Phi_2(y,q)}{\eta(q)^{18}} = -\frac{1}{q\left(y^{\frac{1}{2}}-y^{-\frac{1}{2}}\right)^2}\prod_{n\geq 0}\frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

The main result of that article are Fourier expansions of meromorphic negative index Jacobi forms and in that case Theorem 1.3 of [8] yields

$$\frac{\Phi_2(y,q)}{\eta(q)^{18}} = -\frac{1}{\eta(q)^{24}}\sum_{n\in\mathbb{Z}}\frac{(2n+1)yq^{n(n+1)}}{(1-yq^n)} + \frac{y^2q^{n(n+2)}}{(1-yq^n)^2}.$$

Define the Fourier coefficients $a_{n,h}$ as

$$\frac{\Phi_2(y,q)}{\eta(q)^{18}} = \sum_{\substack{n \in \mathbb{Z} \\ h \geq -1}} a_{n,h} y^n q^h,$$

where $|qy| < 1$ and $|y^{-1}q| < 1$ and also $|y| < 1$. These coefficients can be computed in two fashions and the comparison will yield the theorem. First, using the formula of [41] we get

$$\frac{\Phi_2(y,q)}{\eta(q)^{18}} = -\frac{1}{q\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^2} \prod_{n \geq 0} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}$$

$$= -\frac{1}{q\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^2} \sum_{h \geq 0} \sum_{g=0}^{h} (-1)^g r_{g,h} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2g} q^h$$

$$= -\sum_{h \geq -1} \sum_{g=0}^{h+1} (-1)^g r_{g,h+1} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2(g-1)} q^h$$

$$= -\sum_{h \geq -1} r_{0,h+1} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{-2} q^h - \sum_{h \geq -1} \sum_{g=1}^{h+1} (-1)^g r_{g,h+1} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2(g-1)} q^h$$

$$= -\sum_{h \geq -1} \sum_{n=0}^{\infty} r_{0,h+1} n y^n q^h - \sum_{h \geq -1} \sum_{g=1}^{h+1} (-1)^{g+k} r_{g,h+1} \binom{2(g-1)}{k} y^{g-k-1} q^h.$$

In the last equality we have expanded $\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{-2}$ in the domain $|y| < 1$ and used the binomial formula for $\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^{2(g-1)}$. It follows that

$$-a_{n,h} = n r_{0,h+1} + (-1)^{n+1} \sum_{g=1}^{h+1} (-1)^{g+k} r_{g,h+1} \binom{2(g-1)}{g-n-1}. \tag{5.17}$$

Second, we will compute these coefficient using Theorem 1.3 of [8] together with the expression of $\eta(q)^{-24}$ in terms of the $r_{0,h}$, namely

$$\frac{\Phi_2(y,q)}{\eta(q)^{18}} = -\frac{1}{\eta(q)^{24}} \sum_{n \in \mathbb{Z}} \frac{(2n+1)yq^{n(n+1)}}{(1-yq^n)} + \frac{y^2 q^{n(n+2)}}{(1-yq^n)^2}$$

$$= -\sum_{h \geq 0} r_{0,h} q^h \sum_{n \in \mathbb{Z}} \frac{(2n+1)yq^{n(n+1)}}{(1-yq^n)} + \frac{y^2 q^{n(n+2)}}{(1-yq^n)^2}$$

$$= -\sum_{h \geq 0} r_{0,h} q^h \sum_{n \in \mathbb{Z}} \sum_{m=0}^{\infty} (2n+1)y^{m+1}q^{n(n+m+1)} + (m+1)y^{m+2}q^{n(n+m+2)}$$

and hence the Fourier coefficients are

$$-a_{n,h} = \sum_{s=0}^{\infty} (2s+n) r_{0,h+1-s(s+n)} = n r_{0,h+1} + \sum_{s=1}^{\infty} (2s+n) r_{0,h+1-s(s+n)}. \tag{5.18}$$

So that comparing $a_{n,h-1}$ in (5.17) and (5.18) yields identity (5.16).

6 Modularity

As explained in [14, 18], on physical grounds, the partition functions $Z_{DT}(X, r, \delta; \tau)$ are expected to have modular properties. The goal of this section is to prove modularity for the explicit formulas derived in this paper. In order to fix ideas and notation, the following is a brief self-contained summary of the previous sections.

- Let Λ be a rank $\ell \geq 1$ lattice equipped with an integral even nondegenerate symmetric bilinear form of signature $(1, \ell - 1)$ written as $(\alpha, \alpha') \mapsto \alpha \cdot \alpha'$. This form will be often referred to as the intersection form on Λ because of its geometric origin. Let $(v_i)_{1 \leq i \leq \ell}$ be a basis in Λ. The matrix $(v_i \cdot v_j)$ will be denoted by $M = (M_{ij})$. Let $m = |\det(M)|$.

- Let $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the dual lattice. Using the bilinear form, Λ is identified with a sublattice of $\Lambda^\vee$. For any $r \in \mathbb{Z}$, $r \geq 1$, the quotient $\Lambda^\vee / r\Lambda$ is denoted by G_r. The equivalence class of an element $d \in \Lambda^\vee$ in G_r will be denoted by $[d]_r$. Furthermore for any $r \geq 1$ there is a natural projection $G_r \twoheadrightarrow G_1$. The image of an element $\delta \in G_r$ in G_1 will be denoted by $[\delta]_1$.

- Note that $\Lambda^\vee \subset \Lambda_{\mathbb{Q}}$ and there is a $\mathbb{Q}$-valued nondegenerate symmetric bilinear form pairing on $\Lambda^\vee$ induced by the given intersection form on Λ. In terms of the dual basis $(\check{v}^i)_{1 \leq i \leq \ell}$ this pairing is given by $(\check{v}^i, \check{v}^j) \mapsto (M^{-1})_{ij}$. Moreover, for any $r \geq 1$, the induced form descends to a $\mathbb{Q}/\mathbb{Z}$-valued nondegenerate symmetric bilinear form $(\ ,\ )_r$ on G_r given by

$$([d]_r, [d']_r)_r = \frac{d \cdot d'}{r} \bmod \mathbb{Z}.$$

In particular one obtains a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $\theta_r : G_r \to \mathbb{Q}/\mathbb{Z}$,

$$\theta_r([d]_r) = \frac{d^2}{2r} \bmod \mathbb{Z}. \tag{6.1}$$

- According to [63], the above data determines a representation $\rho_\Lambda : Mp(2, \mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G_1])$, where $Mp(2, \mathbb{Z})$ is the metaplectic double cover of $SL(2, \mathbb{Z})$. The double cover $Mp(2, \mathbb{Z})$ consists of pairs $M, \phi_M(\tau)$ with

$$\phi_M(\tau) = \sqrt{c\tau + d} \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

and is generated by the pairs

$$\widetilde{T} = (T, 1), \qquad \widetilde{S} = (S, \sqrt{\tau}),$$

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which satisfy the relations $(\widetilde{S})^2 = (\widetilde{S}\widetilde{T})^3 = (-I_2, i)$. Then the representation ρ_Λ is defined by

$$\rho_\Lambda(\widetilde{T})(e_\delta) = e^{2\pi i \theta_1(\delta)} e_\delta, \qquad \rho_\Lambda(\widetilde{S})(e_\delta) = \frac{e^{\pi i(\ell-2)/4}}{\sqrt{m}} \sum_{\delta' \in G_1} e^{-2\pi i(\delta,\delta')_1} e_{\delta'} \qquad (6.2)$$

where $(e_\delta)_{\delta \in G_1}$ is a basis of $\mathbb{C}[G_1]$. This is the Weil representation.

- A scaled form of the Weil representation will occur in the following. Namely for any $r \in \mathbb{Z}$, $r \geq 1$, let $\widetilde{\Lambda} \subset \Lambda_\mathbb{R}$ be the sublattice

$$\widetilde{\Lambda} = \{\sqrt{r}\alpha \,|\, \alpha \in \Lambda\}.$$

Clearly, the intersection form on Λ induces an intersection form on $\widetilde{\Lambda}$. The dual lattice $\widetilde{\Lambda}^\vee \subset \Lambda_\mathbb{R}$ is naturally identified with the sublattice

$$\widetilde{\Lambda}^\vee = \{d/\sqrt{r} \,|\, d \in \Lambda^\vee\}.$$

Then note that there is a lattice isomorphism

$$f : \Lambda \xrightarrow{\sim} \widetilde{\Lambda}, \qquad f(\alpha) = \sqrt{r}\alpha$$

which yields an isomorphism of finite abelian groups

$$\phi : G_r \xrightarrow{\sim} \widetilde{\Lambda}^\vee/\widetilde{\Lambda}, \qquad \phi([d]_r) = [d/\sqrt{r}]_1. \qquad (6.3)$$

Using this isomorphism, the Weil representation associated to $\widetilde{\Lambda}$ with the induced bilinear form yields a representation $\rho_{\Lambda,r} : Mp(2,\mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G_r])$ given by

$$\rho_{\Lambda,r}(\widetilde{T})(e_\delta) = e^{2\pi i \theta_r(\delta)} e_\delta, \qquad \rho_{\Lambda,r}(\widetilde{S})(e_\delta) = \frac{e^{\pi i(\ell-2)/4}}{r^{\ell/2}\sqrt{m}} \sum_{\delta' \in G_r} e^{-2\pi i(\delta,\delta')_r} e_{\delta'} \qquad (6.4)$$

where $(e_\delta)_{\delta \in G_r}$ is a basis of $\mathbb{C}[G_r]$.

For a representation ρ of $Mp(2,\mathbb{Z})$ on a group algebra $\mathbb{C}[G]$, We call a function $\Phi(\tau, \bar{\tau}) = \sum_{\delta \in G} \Phi_\delta e_\delta$ on $\mathbb{C}[G]$ vector valued of type ρ and weight $(a, \bar{a})$ if

$$\Phi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^a (c\bar{\tau} + d)^{\bar{a}} \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) \Phi(\tau, \bar{\tau}).$$

See Theorem 4.1 of [6] for Siegel theta functions as examples. The componets Φ_δ then must tansform in the dual representation of ρ and if ρ is unitary and symmetric, as the Weil representation for example, then this is just the complex conjugate.

- For any $r \in \mathbb{Z}$, $r \geq 1$ and for any $\delta \in G_r$ there is a partition function for vertical Donaldson-Thomas invariants of a Λ-polarized K3 pencil given in equation (4.20). This formula is written in terms of the vector valued modular form

$$\widetilde{\Phi}(q) = \sum_{\delta \in G_1} \widetilde{\Phi}_g \delta(q) e_\delta \in \mathbb{C}[[q^{1/2m}]] \otimes \mathbb{C}[G_1]$$

of weight $w = (22 - \ell)/2$ and type ρ_Λ^* which encodes the Nother-Lefschetz numbers of the K3 pencil. Since the geometric details are not important for modularity questions, it will be convenient to consider the following abstract variant of the geometric construction.

Let $\Psi(\tau) = \sum_{\delta \in G_1} \Psi_\delta(\tau) e_\delta$ be a weight $w = -1 - \ell/2$ vector valued modular form of type ρ_Λ^*. Note that for any $k, l \in \mathbb{Z}$, $k, l \geq 1$, there is an injective morphism of abelian groups

$$f_{l,k} : G_l \to G_{kl}, \qquad f_{l,k}([d]_l) = [kd]_{kl}$$

for any $d \in \Lambda^\vee$. Hence any $\delta \in G_{kl}$ such that $k|\delta$ determines a unique element $\eta = f_{l,k}^{-1}(\delta) \in G_l$.

Now, for any $r \in \mathbb{Z}$, $r \geq 1$ and for any $\delta \in G_r$ let

$$Z_{r,\delta}(\tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z},\ k,l \geq 1 \\ r=kl,\ \delta=f_{l,k}(\eta)}} \sum_{s=0}^{l-1} l e^{-2\pi i s l \theta_r(\delta)/r} \Psi_{[\eta]_1} \left(\frac{k\tau + s}{l} \right). \tag{6.5}$$

where $[\eta]_1 \in G_1$ denotes the projection of $\eta \in G_l$ onto G_1.

- Finally, for any $r \in \mathbb{Z}$, $r \geq 1$, let

$$\Theta_r(\tau, \bar\tau) = \sum_{\delta \in G_r} \Theta_{r,\delta}(\tau, \bar\tau) e_\delta,$$

be the vector valued series defined by

$$\Theta_{r,\delta}(\tau, \bar\tau) = \sum_{\alpha \in \Lambda} e^{2\pi i \bar\tau (d+r\alpha)_-^2/2r + 2\pi i \tau (d+r\alpha)_+^2/2r}.$$

Using the isomorphism (6.3), note that

$$\Theta_{r,\delta}(\tau, \bar\tau) = \sum_{\tilde\alpha \in \widetilde\Lambda} e^{2\pi i \bar\tau (\tilde d + \tilde\alpha)_-^2/2 + 2\pi i \tau (\tilde d + \tilde\alpha)_+^2/2},$$

where $\tilde d = d/\sqrt{r}$. The right hand side of the above equation is the Siegel theta function $\Theta_{\tilde\delta}(\tau, \bar\tau)$ for the lattice $\widetilde\Lambda$, with $\tilde\delta = [\tilde d]_1 = \phi(\delta)$. This shows that the series $\Theta_r(\tau, \bar\tau)$ is a vector valued modular form with values in the representation $\rho_{\Lambda,r}^*$ of weight $(1, \ell/2)$ (see section four of [6] for details on vector valued Siegel modular functions.

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- It follows that for any $r \in \mathbb{Z}$, $r \geq 1$, the complex conjugate of the Siegel modular function

$$\Theta_r^*(\tau, \bar{\tau}) = \sum_{\delta \in G_r} \Theta_{r,\delta}^*(\tau, \bar{\tau}) e_\delta,$$

with

$$\Theta_{r,\delta}^*(\tau, \bar{\tau}) = \sum_{\alpha \in \Lambda} e^{-2\pi i \tau (d+r\alpha)_-^2/2r - 2\pi i \bar{\tau}(d+r\alpha)_+^2/2r},$$

is a vector valued modular form with values in the representation $\rho_{\Lambda,r}$ of weight $(\ell/2, 1)$.

In the above framework, the main result proven in this section is:

For any $r \in \mathbb{Z}$, $r \geq 1$, the vector valued series

$$Z_r(\tau) = \sum_{\delta \in G_r} Z_{r,\delta}(\tau) e_\delta$$

is a vector valued modular form of weight $(-1 - \ell/2)$ and type $\rho_{\Lambda,r}^*$.

This confirms the modularity properties of such partition functions derived in [14, 18] on physics grounds. An immediate consequence of this result is that

$$\sum_{\delta \in G_r} Z_{r,\delta}(\tau) \Theta_{r,\delta}^*(\tau, \bar{\tau}),$$

is a non-holomorphic modular form of weight $(-1, 1)$, again as predicted in loc. cit.

Finally, let us note that the space of vector valued modular forms for a given Weil representation and a given weight is always finite-dimensional. The reason is that all components are modular for $\Gamma_0(N)$ for some N with a Dirichlet character. The space of such modular forms is however finite-dimensional for a given weight (by Riemann-Roch). Concrete dimensions are explicitely stated for the case of weight at least two in [11]. Since the map $f \mapsto f\Delta$ (Δ the unique cusp form for $SL(2, \mathbb{Z})$ of weight 12) from weight k to weight $k+12$ modular forms is injective these dimensions give upper bounds for the cases of weight smaller than two.

6.1 Proof

The goal of this section is to prove the above modularity statement. The standard notation $e(x) = e^{2\pi i x}$ will be used for brevity. To prove modularity, one needs to show that

$$Z_{r,\delta}\left(-\frac{1}{\tau}\right) = \frac{\tau^{-1-\ell/2} e\left(\frac{\ell-2}{8}\right)}{\sqrt{m} r^{\ell/2}} \sum_{\delta' \in G_r} e(-(\delta, \delta')_r) Z_{r,\delta'}(\tau). \tag{6.6}$$

The first step in the proof is to rewrite the series $Z_{r,\delta}(\tau)$ as follows. Note that for any $k, l \in \mathbb{Z}$, $k, l \geq 1$, there are morphisms of $\mathbb{Z}$-modules

$$f_{l,k} : \Lambda/l\Lambda \longrightarrow \Lambda/(kl)\Lambda, \qquad g_{k,l} : \Lambda/(kl)\Lambda \longrightarrow \Lambda/l\Lambda$$

where

$$f_{l,k}([\alpha]_l) = [k\alpha]_{kl}, \qquad g_{k,l}([\alpha]_{kl}) = [\alpha]_l$$

for all $\alpha \in \Lambda$. Note also that $f_{l,k}$ is injective and $g_{k,l}$ is surjective. For any $\delta \in \Lambda/(kl)\Lambda$ let $\delta_l = g_{k,l}(\delta)$. Then $Z(r, \delta)$ can be written as

$$Z_{r,\delta}(\tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z}_{\geq 1},\, kl=r \\ \delta = f_{l,k}(\eta)}} \sum_{s=0}^{l-1} le(-s\theta_l(\eta))\Psi_{\eta_1}\left(\frac{k\tau + s}{l}\right). \tag{6.7}$$

At this point one needs to distinguish between the $s = 0$ and $s > 0$ cases. For $s > 0$, let $p = \gcd(s, l)$ and $\tilde{s} = s/p$, $\tilde{l} = l/p$. Note that $0 < \tilde{s} < \tilde{l}$. Then (6.7) can be rewritten as

$$Z_{r,\delta}(\tau) = \frac{1}{2r^2} \sum_{\substack{k,l \in \mathbb{Z}_{\geq 1},\, kl=r \\ \delta = f_{l,k}(\eta)}} l\Psi_{\eta_1}\left(\frac{k\tau}{l}\right)$$

$$+ \frac{1}{2r^2} \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}^{\geq 1},\, k\tilde{l}p=r \\ \delta = f_{p\tilde{l},k}(\eta)}} p\tilde{l} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}}))\Psi_{\eta_1}\left(\frac{k\tau + p\tilde{s}}{p\tilde{l}}\right). \tag{6.8}$$

Using the above formula, one can calculate $Z_{r,\delta}(-1/\tau)$ as follows. For the $s = 0$ terms note that

$$\Psi_{\eta_1}\left(-\frac{k}{l\tau}\right) = \frac{l^{-1-\ell/2}k^{1+\ell/2}\tau^{-1-\ell/2}e\left(\frac{\ell-2}{8}\right)}{\sqrt{m}} \sum_{\mu \in G_1} e(-(\eta_1, \mu)_1)\Psi_\mu\left(\frac{l\tau}{k}\right). \tag{6.9}$$

since $\Psi(\tau)$ is a vector valued modular form of weight $(-1 - \ell/2)$ and type ρ_Λ^*. In order to evaluate the $s \geq 1$ terms, one has to compute

$$\Psi_{\eta_1}\left(-\frac{k}{p\tilde{l}\tau} + \frac{\tilde{s}}{\tilde{l}}\right) \tag{6.10}$$

using the modular properties of Ψ. Let

$$\tau' = \frac{p}{k\tilde{l}}\tau + \frac{s'}{\tilde{l}}, \tag{6.11}$$

where $s' \in \{0, \ldots, l-1\}$ is uniquely defined by the requirement that $s'\tilde{s} = -1 \mod \tilde{l}$. Then, it is easy to check that

$$-\frac{k}{p\tilde{l}\tau} + \frac{\tilde{s}}{\tilde{l}} = \frac{a\tau' + b}{c\tau' + d}, \tag{6.12}$$

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for the $SL(2, \mathbb{Z})$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tilde{s} & -\frac{\tilde{s}s'+1}{\tilde{l}} \\ \tilde{l} & -s' \end{pmatrix}. \tag{6.13}$$

Now note that Shintani's formula [53], reviewed in Appendix A, yields an identity:

$$\Psi_{\eta_1}\left(-\frac{k}{p\tilde{l}\tau} + \frac{\tilde{s}}{\tilde{l}}\right) = \frac{p^{-1-\ell/2}\tau^{-1-\ell/2}k^{1+\ell/2}\tilde{l}^{-\ell/2}e\left(\frac{\ell-2}{8}\right)}{\sqrt{m}}\sum_{\sigma \in G_1} \bar{h}_{-s',\tilde{s}}(\eta_1, \sigma)\Psi_\sigma\left(\frac{p}{k\tilde{l}}\tau + \frac{s'}{\tilde{l}}\right). \tag{6.14}$$

where $\bar{h} : G_1 \times G_1 \to \mathbb{C}$ is a function defined as follows. Pick any lift $\mu \in G_{\tilde{l}}$ of σ, such that $\mu_1 = \sigma$. Then

$$\bar{h}_{-s',\tilde{s}}(\eta_1, \sigma) = \sum_{u \in \Lambda/\tilde{l}\Lambda} e\left(-\frac{s'}{2}(\mu + u, \mu + u)_{\tilde{l}} - (\mu + u, \eta_{\tilde{l}})_{\tilde{l}} + \frac{\tilde{s}}{2}(\eta_{\tilde{l}}, \eta_{\tilde{l}})_{\tilde{l}}\right).$$

As shown in Appendix A, the right hand side of the above equation does not depend on the chosen reresentative μ.

Combining (6.9) and (6.14), one gets

$$Z_{r,\delta}\left(-\frac{1}{\tau}\right) = \frac{\tau^{-1-\ell/2}e\left(\frac{\ell-2}{8}\right)}{2r^2\sqrt{m}}\left(S_0(\tau) + S_{\geq 1}(\tau)\right), \tag{6.15}$$

with

$$S_0(\tau) = \sum_{\substack{k,l \in \mathbb{Z}_{\geq 1}, \, kl=r \\ \delta = f_{l,k}(\eta)}} l^{-\ell/2}k^{1+\ell/2}\sum_{\sigma \in G_1} e(-(\eta_1, \sigma)_1)\Psi_\sigma\left(\frac{l\tau}{k}\right), \tag{6.16}$$

and

$$S_{\geq 1}(\tau) = \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1}, \, k\tilde{l}p=r \\ \delta = f_{p\tilde{l},k}(\eta)}} p^{-\ell/2}\tilde{l}^{1-\ell/2}k^{1+\ell/2}\sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}}))\sum_{\sigma \in G_1} \bar{h}_{-s',\tilde{s}}(\eta_1, \sigma)\Psi_\sigma\left(\frac{p}{k\tilde{l}}\tau + \frac{s'}{\tilde{l}}\right).$$

Now note that

$$e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}}))\bar{h}_{-s',\tilde{s}}(\eta_1, \sigma) = \sum_{u \in \Lambda/\tilde{l}\Lambda} e\left(-\frac{s'}{2}(\mu + u, \mu + u)_{\tilde{l}} - (\mu + u, \eta_{\tilde{l}})_{\tilde{l}}\right)$$

where $\mu \in G_{\tilde{l}}$ is an arbitrary lift of $\sigma \in G_1$. As observed in Appendix A, the function $h_{\eta_{\tilde{l}}, -s'} : G_{\tilde{l}} \to \mathbb{C}$,

$$h_{\eta_{\tilde{l}}, -s'}(\mu) = \sum_{u \in \Lambda/\tilde{l}\Lambda} e\left(-\frac{s'}{2}(\mu + u, \mu + u)_{\tilde{l}} - (\mu + u, \eta_{\tilde{l}})_{\tilde{l}}\right) \tag{6.17}$$

is invariant under shifts $\mu \mapsto \mu + x$, $x \in \Lambda/\tilde{l}\Lambda$. Hence it descends to a function $\bar{h}_{\eta_{\tilde{l}}, -s'} : G_1 \to \mathbb{C}$. Therefore

$$S_{\geq 1}(\tau) = \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}^{\geq 1}, \, k\tilde{l}p=r \\ \delta = f_{p\tilde{l},k}(\eta)}} p^{-\ell/2}\tilde{l}^{1-\ell/2}k^{1+\ell/2}\sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}}\sum_{\sigma \in G_1} \bar{h}_{\eta_{\tilde{l}}, -s'}(\sigma)\Psi_\sigma\left(\frac{p}{k\tilde{l}}\tau + \frac{s'}{\tilde{l}}\right). \tag{6.18}$$

Returning to equation (6.6), one now has to prove that

$$\frac{1}{r^{\ell/2}} \sum_{\delta' \in G_r} e(-(\delta, \delta')_r) Z_{r,\delta'}(\tau) = \frac{1}{2r^2} \left(S_0(\tau) + S_{\geq 1}(\tau) \right). \tag{6.19}$$

First, using (6.8) the left-hand-side can be written as

$$\frac{1}{r^{\ell/2}} \sum_{\delta' \in G_r} e(-(\delta, \delta')_r) Z_{r,\delta'}(\tau) = \frac{1}{2r^2 r^{\ell/2}} \sum_{\delta' \in G_r} e(-(\delta, \delta')_r) \left(\sum_{\substack{k,l \in \mathbb{Z}_{\geq 1},\, kl=r \\ \delta' = f_{l,k}(\eta)}} l \Psi_{\eta_1} \left(\frac{k\tau}{l} \right) \right.$$

$$\left. + \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1},\, k\tilde{l}p=r \\ \delta' = f_{p\tilde{l},k}(\eta)}} p\tilde{l} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}})) \Psi_{\eta_1} \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}} \right) \right)$$

$$= \frac{1}{2r^2} \left(Q_0(\tau) + Q_{\geq 1}(\tau) \right),$$

where

$$Q_0(\tau) = \frac{1}{r^{\ell/2}} \sum_{\substack{k,l \in \mathbb{Z}_{\geq 1},\, kl=r \\ \eta \in G_l}} l e(-(\delta_l, \eta)_l) \Psi_{\eta_1} \left(\frac{k\tau}{l} \right), \tag{6.20}$$

and

$$Q_{\geq 1}(\tau) = \frac{1}{r^{\ell/2}} \sum_{\eta \in G_l} e(-(\delta_l, \eta)_l) \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1},\, k\tilde{l}p=r}} p\tilde{l} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}_1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}})) \Psi_{\eta_1} \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}} \right). \tag{6.21}$$

It will be shown below that $Q_0(\tau) = S_0(\tau)$ and $Q_{\geq 1}(\tau) = S_{\geq 1}(\tau)$.

In order to compute $Q_0(\tau)$ note that there is an exact sequence of finite abelian groups

$$0 \to \Lambda/l\Lambda \to G_l \to G_1 \to 0.$$

Given any element $\sigma \in G_1$, let $\mu \in G_l$ denote an arbitrary lift of σ. Then one has

$$\sum_{\substack{\eta \in G_l \\ \eta_1 = \sigma}} e(-(\delta_l, \eta)_l) = \sum_{u \in \Lambda/l\Lambda} e(-(\delta_l, \mu + u)_l) = e(-(\delta_l, \mu)_l) \sum_{u \in \Lambda/l\Lambda} e(-(\delta_l, u)_l).$$

Now note that

$$\sum_{u \in \Lambda/l\Lambda} e(-(\delta_l, u)_l) = \begin{cases} l^\ell & \text{if } \delta = f_{k,l}(\xi) \text{ for some } \xi \in G_k, \\ 0 & \text{otherwise.} \end{cases} \tag{6.22}$$

Moreover if $\delta = f_{k,l}(\xi)$ for some $\xi \in G_k$, then $e(-(\delta_l, \mu)_l) = e(-(\xi_1, \mu_1)_1) = e(-(\xi_1, \sigma)_1)$. Therefore,

$$Q_0(\tau) = \sum_{\substack{k,l \in \mathbb{Z}_{\geq 1},\, kl=r \\ \delta = f_{k,l}(\xi)}} k^{-\ell/2} l^{1+\ell/2} \sum_{\sigma \in G_1} e(-(\xi_1, \sigma)_1) \Psi_\sigma \left(\frac{k\tau}{l} \right). \tag{6.23}$$

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Comparing with $S_0(\tau)$ in (6.16), exchanging the summation variables $l \leftrightarrow k$ and the symbols $\xi \leftrightarrow \eta$ yields

$$Q_0(\tau) = S_0(\tau), \tag{6.24}$$

as claimed above.

The computation of $Q_{\geq 1}(\tau)$ is similar. Recall that

$$Q_{\geq 1}(\tau) = \frac{1}{r^{\ell/2}} \sum_{\eta \in G_l} e(-(\delta_l, \eta)_l) \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1}, k\tilde{l}p = r}} p\tilde{l} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}})) \Psi_{\eta_1} \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}} \right). \tag{6.25}$$

The right hand side of equation (6.25) can be written as

$$\frac{1}{r^{\ell/2}} \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1}, k\tilde{l}p = r}} p\tilde{l} \sum_{\eta \in G_l} e(-(\delta_l, \eta)_l) A(\eta_{\tilde{l}})$$

where

$$A(\eta_{\tilde{l}}) = \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} e(-\tilde{s}\theta_{\tilde{l}}(\eta_{\tilde{l}})) \Psi_{\eta_1} \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}} \right)$$

depends only on $\eta_{\tilde{l}}$. In order to compute the sum over $\eta \in G_l$ with fixed projection $\gamma = \eta_{\tilde{l}} \in G_{\tilde{l}}$, note that there is an exact sequence of finite abelian groups

$$0 \to \Lambda/p\Lambda \xrightarrow{i} G_l \to G_{\tilde{l}} \to 0.$$

Given an element $v \in \Lambda/p\Lambda$, one has $i(v) = [\tilde{l}\alpha]_l$ where $\alpha \in \Lambda$ is an arbitrary representative of v. The right hand side, $[\tilde{l}\alpha]_l \in G_l$ is clearly independent of the choice of α. Then let $\gamma_0 \in G_l$ denote an arbitrary representative of γ and note that

$$\sum_{\substack{\eta \in G_l \\ \eta_{\tilde{l}} = \gamma}} e(-(\delta_l, \eta)_l) = e(-(\delta_l, \gamma_0)_l) \sum_{v \in \Lambda/p\Lambda} e(-(\delta_p, v)_p) \tag{6.26}$$

since $l = p\tilde{l}$. As before,

$$\sum_{v \in \Lambda/p\Lambda} e(-(\delta_p, v)_p) = \begin{cases} p^\ell & \text{if } \delta = f_{k\tilde{l},p}(\xi) \text{ for some } \xi \in G_{k\tilde{l}}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.27}$$

Moreover, if $\delta = f_{k\tilde{l},p}(\xi)$ then

$$e(-(\delta_l, \gamma_0)_l) = e(-(\xi_{\tilde{l}}, \gamma)_{\tilde{l}}).$$

Therefore,

$$Q_{\geq 1}(\tau) = \sum_{\substack{k,\tilde{l},p \in \mathbb{Z}_{\geq 1}, k\tilde{l}p = r \\ \delta = f_{k\tilde{l},p}(\xi)}} k^{-\ell/2} p^{1+\ell/2} \tilde{l}^{1-\ell/2} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s},\tilde{l})=1}} \sum_{\gamma \in G_{\tilde{l}}} e(-(\xi_{\tilde{l}}, \gamma)_{\tilde{l}} - \tilde{s}\theta_{\tilde{l}}(\gamma)) \Psi_{\gamma_1} \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}} \right). \tag{6.28}$$

Finally, using again the exact sequence

$$0 \to \Lambda/\tilde{l}\Lambda \to G_{\tilde{l}} \to G_1 \to 0,$$

note that

$$\sum_{\substack{\gamma \in G_{\tilde{l}} \\ \gamma_1 = \rho}} e(-(\xi_{\tilde{l}}, \gamma)_{\tilde{l}} - \tilde{s}\theta_{\tilde{l}}(\gamma)) = \sum_{v \in \Lambda/\tilde{l}\Lambda} e\left(-\frac{\tilde{s}}{2}(\rho_0 + v, \rho_0 + v)_{\tilde{l}} - (\rho_0 + v, \xi_{\tilde{l}})_{\tilde{l}}\right).$$

where $\rho_0 \in G_{\tilde{l}}$ is an arbitrary lift of $\rho \in G_1$. Therefore

$$\sum_{\substack{\gamma \in G_{\tilde{l}} \\ \gamma_1 = \rho}} e(-(\xi_{\tilde{l}}, \gamma)_{\tilde{l}} - \tilde{s}\theta_{\tilde{l}}(\gamma)) = h_{\xi_{\tilde{l}}, -\tilde{s}}(\rho_0)$$

where $h_{\xi_{\tilde{l}}, -\tilde{s}} : G_{\tilde{l}} \to \mathbb{C}$ is the function defined in (6.17), which descends to the function $\bar{h}_{\xi_{\tilde{l}}, -\tilde{s}} : G_1 \to \mathbb{C}$. Hence one obtains

$$\sum_{\substack{\gamma \in G_{\tilde{l}} \\ \gamma_1 = \rho}} e(-(\xi_{\tilde{l}}, \gamma)_{\tilde{l}} - \tilde{s}\theta_{\tilde{l}}(\gamma)) = \bar{h}_{\xi_{\tilde{l}}, -\tilde{s}}(\rho),$$

independent of the choice of ρ_0. In conclusion the final expression for $Q_{\geq 1}(\tau)$ is

$$Q_{\geq 1}(\tau) = \sum_{\substack{k, \tilde{l}, p \in \mathbb{Z}_{\geq 1}, \, k\tilde{l}p = r \\ \delta = f_{k\tilde{l}, p}(\xi)}} k^{-\ell/2} p^{1+\ell/2} \tilde{l}^{1-\ell/2} \sum_{\substack{1 \leq \tilde{s} \leq \tilde{l}-1 \\ (\tilde{s}, \tilde{l}) = 1}} \sum_{\rho \in G_1} \bar{h}_{\xi_{\tilde{l}}, -\tilde{s}}(\rho) \Psi_\rho \left(\frac{k\tau + p\tilde{s}}{p\tilde{l}}\right). \tag{6.29}$$

Comparing with (6.18), exchanging the summation variables $p \leftrightarrow k$, $s' \leftrightarrow \tilde{s}$, and the symbols $\xi \leftrightarrow \eta$ yields

$$Q_{\geq 1}(\tau) = S_{\geq 1}(\tau), \tag{6.30}$$

concluding the proof.

A   Shintani's formula

This section is a brief review of Shintani's explicit formula [53, Prop. 1.6] for the matrix elements of the Weil representation. As in Section 6, consider a lattice Λ equipped with an integral even nondegenerate symmetric bilinear form. Let ℓ denote the rank of Λ and m denote the absolute value of the determinant of the given bilinear form. According to [63] such a lattice determines a representation

$$\rho_\Lambda : Mp(2, \mathbb{Z}) \to \text{End}(\mathbb{C}[G_1])$$

where $Mp(2,\mathbb{Z})$ is the metaplectic double cover of $SL(2,\mathbb{Z})$ and $G_1 = \Lambda^\vee/\Lambda$. The action of the standard generators of the metaplectic group is given in (6.2). Shintani's result [53, Prop. 1.6] provides an explicit formula for the matrix elements associated to an arbitrary element

$$\tilde{\sigma} = (\sigma, \sqrt{c\tau + d}), \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of the metaplectic group with $c > 0$. In order to write this formula in detail consider first the function

$$f_{a,d} : \Lambda^\vee/c\Lambda \times \Lambda^\vee/c\Lambda \times \Lambda/c\Lambda \to \mathbb{C}$$

$$f_{a,d}(\gamma,\zeta,u) = e\left(\frac{a}{2}(\gamma+u,\gamma+u)_c - (\gamma+u,\zeta)_c + \frac{d}{2}(\zeta,\zeta)_c\right)$$

Next define a function

$$g_{a,d} : \Lambda^\vee/c\Lambda \times \Lambda^\vee/c\Lambda \to \mathbb{C}$$

$$g_{a,d}(\gamma,\zeta) = \sum_{u \in \Lambda/c\Lambda} f(\gamma,\zeta,u). \tag{A.1}$$

Then one checks as follows that the function $g_{a,d}$ satisfies the invariance conditions

$$g_{a,d}(\gamma+x,\zeta) = g_{a,d}(\gamma,\zeta+y) = g_{a,d}(\gamma,\zeta)$$

for any $(\gamma,\zeta) \in \Lambda^\vee/c\Lambda \times \Lambda^\vee/c\Lambda$ and any $(x,y) \in \Lambda/c\Lambda \times \Lambda/c\Lambda$.

Invariance under $\gamma \mapsto \gamma + x$ is easily proven by changing the summation variable u to $u - x$ in the right hand side of (A.3).

Invariance under $\zeta \mapsto \zeta + y$ requires more work. Recall that the pairing $(\ ,\ )_c$ takes integral values on $\Lambda^\vee/c\Lambda \times \Lambda/c\Lambda$ and restricts to an even integral form on $\Lambda/c\Lambda \times \Lambda/c\Lambda$. Then, using the basic relation $ad - bc = 1$, one has the folllowing sequence of congruences mod $2\mathbb{Z}$:

$$a(\gamma+u,\gamma+u)_c - 2(\gamma+u,\zeta+y)_c + d(\zeta+y,\zeta+y)_c \equiv$$
$$a(\gamma+u,\gamma+u)_c - 2ad(\gamma+u,y)_c + ad^2(y,y)_c - 2(\gamma+u-dy,\zeta)_c + d(\zeta,\zeta)_c \equiv \tag{A.2}$$
$$a(\gamma+u-dy,\gamma+u-dy)_c - 2(\gamma+u-dy,\zeta)_c + d(\zeta,\zeta)_c.$$

Now invariance reduces again to a shift $u \mapsto u + dy$ of the summation variable in (A.3).

In conclusion $g_{a,d}$ descends to a function

$$\bar{g}_{a,d} : \Lambda^\vee/\Lambda \times \Lambda^\vee/\Lambda \to \mathbb{C}$$

Then Shintani's formula reads

$$\rho_\Lambda(\tilde{\sigma})_{\delta_1,\delta_2} = \frac{e\left(\frac{\ell-2}{8}\right)}{c^{\ell/2}m^{1/2}}\, \bar{g}_{a,d}(\delta_1,\delta_2).$$

For the modularity proof in Section 6 we need a slightly different presentation of Shintani's formula. Let

$$f'_{a,d} : \Lambda^\vee/c\Lambda \times \Lambda^\vee/c\Lambda \times \Lambda/c\Lambda \to \mathbb{C}$$

be the function defined by

$$f'_{a,d}(\gamma, \zeta, u) = e\left(\frac{a}{2}(\gamma, \gamma)_c - (\gamma, \zeta + u)_c + \frac{d}{2}(\zeta + u, \zeta + u)_c\right)$$

Let

$$h_{a,d} : \Lambda^\vee/c\Lambda \times \Lambda^\vee/c\Lambda \to \mathbb{C}$$

be defined by

$$h_{a,d}(\gamma, \zeta) = \sum_{u \in \Lambda/c\Lambda} f'_{a,d}(\gamma, \zeta, u). \tag{A.3}$$

Then by analogy with the above argument for $g_{a,d}$, the function $h_{a,d}$ also descends to a function

$$\bar{h}_{a,d} : \Lambda^\vee/\Lambda \times \Lambda^\vee/\Lambda \to \mathbb{C}.$$

Furthermore it will be shown below that

$$h_{a,d}(\gamma, \zeta) = g_{a,d}(\gamma, \zeta)$$

for any (γ, ζ).

By analogy with equation (A.2) one has

$$a(\gamma, \gamma)_c - 2(\gamma, \zeta + u)_c + d(\zeta + u, \zeta + u)_c \equiv a(\gamma - du, \gamma - du)_c - 2(\gamma - du, \zeta)_c + d(\zeta, \zeta)_c \mod 2\mathbb{Z}.$$

Hence

$$h_{a,d}(\gamma, \zeta) = \sum_{u \in \Lambda/c\Lambda} f_{a,d}(\gamma, \zeta, -du).$$

Now note that d, c are coprime since $ad - bc = 1$. Since $\Lambda/c\Lambda \simeq (\mathbb{Z}/c\mathbb{Z})^{\times \ell}$, this implies that multiplication by d gives an isomorphism $\Lambda/c\Lambda \xrightarrow{d} \Lambda/c\Lambda$. Therefore

$$\sum_{u \in \Lambda/c\Lambda} f_{a,d}(\gamma, \zeta, -du) = \sum_{u \in \Lambda/c\Lambda} f_{a,d}(\gamma, \zeta, u),$$

which proves the claim.

Finally, for any fixed element $\gamma \in \Lambda^\vee/c\Lambda$, define

$$h_{\gamma,d} : \Lambda^\vee/c\Lambda \to \mathbb{C}$$

$$h_{\gamma,d}(\zeta) = \sum_{u \in \Lambda/c\Lambda} e\left(\frac{d}{2}(\zeta + u, \zeta + u)_c - (\zeta + u, \gamma)_c\right)$$

Then it follows again that $h_{\gamma,d}(\zeta + y) = h_{\gamma,d}(\zeta)$ for any $\zeta \in \Lambda^\vee/c\Lambda$, $y \in \Lambda/c\Lambda$. Hence $h_{\gamma,d}$ descends to a function $\bar{h}_{\gamma,d} : \Lambda^\vee/\Lambda \to \mathbb{C}$.

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