MATH 215

Calculus IV

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These are notes for MATH 215 offered at the University of Alberta, which is the fourth calculus course in the calculus sequence. MATH 215 is devoted to vector calculus.

How to use these notes

These notes are lectures notes for MATH 215. They are structured to accompany the lectures offered at the University of Alberta.

Each 50 minute lecture in the course corresponds to a subsection in the notes (except for Subsection 6.5 which is a summary section). Further, each subsection is structured as follows:

- A brief summary of the content of the subsection;
- The learning objectives of the subsection;
- The course notes for the topic of the subsection, with worked out examples;
- Exercises (with full solutions) on the topic of the subsection.

These notes are meant to be supplemented with the in-class notes annotated during the lectures, which are numbered to match the section numbers in these notes. The in-class notes will be posted on eClass. In particular, these notes do not include many figures -- these will be provided in the accompanying in-class notes. More figures will be added to these notes in the future.

Please do not hesitate to let me know if you have comments or find typos/mistakes at vincent.bouchard@ualberta.ca.

Let us now delve into the beauty and depth of vector calculus!

Supplementary material and references

These course notes for MATH 215 were written with the help of various other notes, including the following main references. Those references may also be very helpful as supplementary material.

- CLP4, the Calculus IV open textbook from UBC. Focuses on vector calculus, but takes a traditional approach, relegating differential forms to the optional Section 4.7.
- Lecture notes on differential forms and vector calculus by Donu Arapura from Purdue University. A fairly short introduction to the topic of vector calculus using differential forms.
- Lecture notes by George Peschke from the University of Alberta, available on eClass. Approaches the topic in a fairly similar way to these course notes, but with more rigor and generality.
- APEX Calculus (version with videos from Sean Fitzpatrick, University of Lethbridge). Open calculus textbook with videos by Sean Fitzpatrick from the University of Lethbridge. Chapter 15 is about vector calculus, taking a traditional approach (without even mentioning differential forms).
- Multivariable Calculus, by James Stewart, Daniel Clegg, and Saleem Watson (unfortunately not open nor free). Chapter 16 is about vector calculus, following a traditional approach without mention of differential forms.

Contents

Chapter 1

A preview of vector calculus

1.1 A preview of vector calculus

1.1.1 Motivation

Have you ever wondered why, in the definition of definite integrals

$$\int_a^b f(x)\ dx,$$

there is always a "dx" inside the integrand? What does it mean, and why is it there? When you were introduced to definite integrals in Calculus I, they were defined as limits of Riemann sums:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^n f(a + i\Delta x)\Delta x,$$

where Δx is the width of the rectangles in the Riemann sum. It was then argued that in the limit as $n \to \infty$, the width of the rectangles goes to zero, and somehow Δx becomes "dx", which is some sort of infinitesimal width of the rectangles. But what does that mean, really? What is this "dx" thing inside the definite integral?

And it's not like we can just forget about it, even though many students actually do in first year. :-) We all know how important dx is: just think of substitution for definite integrals. Without the dx, substitution would fail miserably. It must be there. Why?

Things become even more interesting with double and triple integrals:

$$\iint_D f\ dA, \qquad \iiint_D f\ dV,$$

with $dA = dxdy$ and $dV = dxdydz$ in Cartesian coordinates. What are these objects dA and dV, and why must they be there in the integrand?

What we will do in this course is provide an answer to this question. Our goal is to define a unified theory of integration for curves, surfaces, and volumes, which will make the appearance of dx, dA, and dV natural. To achieve this, we must define a new type of objects that will play the role of integrands: those are called differential forms, or n-forms. More specifically, one-forms are objects that can be integrated over curves, two-forms over surfaces, and three-forms over volumes. In other words, it was all a big lie: *what you should be integrating is not functions, but rather differential forms!*

But before we start, we can already identify two key guiding principles for the construction, using what we already know about integration.

- *We want our theory to be "reparametrization-invariant".* Consider a definite integral in one dimension. Instead of writing $\int_a^b f(x)\, dx$, we would like to write something like $\int_C \omega$, where C stands for a curve, and ω for the integrand, which will be called a "one-form". (We will also allow C to be a curve in $\mathbb{R}^2$ and $\mathbb{R}^3$). More precisely, to make sense of this expression, we will need C to be a parametric curve. However, we want $\int_C \omega$ to be defined intrinsically in terms of the geometry of the curve itself: we do not want the integral to depend on the choice of parametrization. This is key. This constraint will be satisfied if the integrand ω transforms in a specific way under reparametrizations of the curve; in one dimension this will reproduce the substitution formula for definite integrals.

- *We want our theory to be "oriented".* Consider $\int_C \omega$ as above. As mentioned, to make sense of this expression we will work with a parametric curve C. But once we parametrize a curve, we introduce a choice of "orientation": we select one of the two endpoints as the starting point, and we introduce a "direction of travel along the curve" (the orientation is given by the direction of the tangent vector, or velocity, for a parametric curve). If we do a reparametrization of the curve that reverses the orientation, should the integral remain invariant? The answer is no! We see this directly in basic calculus: we know that $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$. If we interpret the first integral as being over the interval $[a, b]$ with direction of travel from a to b, and the second integral as being over the same interval but with the reverse orientation, then we see that exchanging the direction of travel over the interval $[a, b]$ changes the sign of the integral.

Putting these two properties together, we see that if C and C' are two parametrizations of the same curve that preserve the orientation, we must have $\int_C \omega = \int_{C'} \omega$, while if they reverse the orientation, we must have $\int_C \omega = -\int_{C'} \omega$. This is what it means to say that the theory should be "oriented" and "reparametrization-invariant".

This brief exposition focused on curves, but following these two guiding principles we will develop a unified theory of oriented integrals not only over curves, but over surfaces and volumes as well, using the machinery of differential forms. Along the way we will discover the beautiful intricacies of vector calculus, culminating with the very important Stokes' Theorem. Let us embark on this journey together!

1.1.2 Vector calculus and differential forms: two sides of the same coin

What we will study in this course is known as vector calculus. There are two main approaches to vector calculus. On the one hand, there is the "traditional" approach, which involves concepts such as gradient, curl, div, etc. It relies heavily on the geometry of $\mathbb{R}^3$, and is very explicit. But at first it seems like a complicated amalgation of strange constructions and definitions that satisfy all kinds of intricate identities. My recollections of learning vector calculus is that it involves many formulae that appear to come out of nowhere and that one needs to learn by heart. Not fun.

On the other hand, there is the "modern" approach, pioneered by Cartan, which relies on the definition of differential forms. This approach is a little more abstract, but is much more unified and elegant. It brings together all the concepts of vector calculus in a unified formalism, from which all the identities and formulae come out naturally. It also does not rely on the geometry of $\mathbb{R}^3$, and is naturally generalized to $\mathbb{R}^n$ (even though we will focus on $\mathbb{R}^3$ in this course). I remember this feeling of "ah, now this all makes sense!" when I learned differential forms later on in my studies.

In this course we will take the perhaps non-traditional approach of introducing vector calculus directly through the unified formalism of differential forms, guided by the exposition of the previous subsection. The challenge is to make the concepts accessible to second-year students, stripping them down from the fancier definitions of differential geometry. But I truly believe that this is possible, and that it will make the whole theory of vector calculus much more interesting and unified, and less reliant on brute force memorization.

But, at the same time, it is important for students to be fluent with the traditional concepts of vector calculus. Indeed, students who will study topics like fluid mechanics, electromagnetism, applied mathematics, etc. will repeadtedly encounter vector calculus, usually expressed in the traditional language. Moreover, traditional concepts such as grad, div, curl, are often useful for explicit calculations. So in this course we will translate all concepts from differential forms to standard vector calculus every step of the way.

In the end, the goal is for students to be fluent with both approaches: to see the beauty and elegance of differential forms, and to be able to use the traditional approach for explicit calculations.

Chapter 2

One-forms and vector fields

In this section we study one-forms, which will become the objects that can be integrated along curves. The counterpart to one-forms in traditional vector calculus is vector fields.

2.1 One-forms and vector fields

We define the concept of differential one-forms (or more simply one-forms), which will become the objects that can be integrated along curves in our theory of integration. We also define vector fields, and show that the two are closely connected: given a one-form, there is an associated vector field, and vice-versa.

Objectives

You should be able to:

- Define one-forms and vector fields in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.
- Translate between one-forms and vector fields.
- Visualize a vector field in $\mathbb{R}^2$ by plotting vectors on a grid.

2.1.1 One-forms

Let us start by defining one-forms in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Definition 2.1.1 One-forms. A **differential one-form** (or simply **one-form**) on an open interval (or union of open intervals) $U \subseteq \mathbb{R}$ is an expression of the form

$$\omega = f(x)\ dx$$

with $f : U \to \mathbb{R}$ a function with continuous derivatives (we say that the function f is "smooth", or C^∞, on U).

A one-form on an open subset[1] $U \subseteq \mathbb{R}^2$ is an expression of the form

$$\omega = f(x, y)\ dx + g(x, y)\ dy$$

with $f, g : U \to \mathbb{R}$ functions with continuous partial derivatives (again, we say that they are smooth, or C^∞, on U).

A one-form on an open subset $U \subseteq \mathbb{R}^3$ is an expression of the form

$$\omega = f(x, y, z)\ dx + g(x, y, z)\ dy + h(x, y, z)\ dz$$

for smooth functions $f, g, h : U \to \mathbb{R}$. ♢

For the time being, you can think of the objects dx, dy and dz as placeholders; they will be given an interpretation shortly. We remark here that the definition can easily be extended to $\mathbb{R}^n$ for any positive integer n, but for the sake of simplicity, we will focus only on $n = 1, 2, 3$ in this course.

We note here two obvious properties of one-forms, which follow from the definition.

1. If ω and η are one-forms on U, then the sum $\omega + \eta$ is also a one-form on U.
2. If ω is a one-form on U and f a smooth function on U, then the product $f\omega$ is also a one-form on U.

2.1.2 Vector fields

There is a closely related concept in vector calculus, which is called a vector field.

Definition 2.1.2 Vector fields. A **vector field** on an open subset $U \subseteq \mathbb{R}^n$ is a function $\mathbf{F} : U \to \mathbb{R}^n$. In other words, it is a rule which assigns to each point in $U \subseteq \mathbb{R}^n$ a vector with n components. ♢

For example, we can write a vector field $\mathbf{F}$ in $U \subseteq \mathbb{R}^2$ as

$$\mathbf{F}(x, y) = (f_1(x, y), f_2(x, y)),$$

where f_1, f_2 are functions $f_1, f_2 : U \to \mathbb{R}$. We call these functions the **component functions** of the vector field. We say that the vector field is **smooth** if the component functions are smooth.

There are many examples of vector fields in physics: think of the velocity vector field of a moving fluid, or the gravitational force vector field produced by a mass.

Looking at the two definitions above, we see a natural correspondence between one-forms and vector fields.

Principle 2.1.3 Correspondence between one-forms and vector fields. *Given a one-form $\omega = f\ dx + g\ dy + h\ dz$, we can define an associated smooth vector field with component functions (f, g, h). Conversely, given a smooth vector field $\mathbf{F} = (f, g, h)$, we can define an associated one-form $\omega = f\ dx + g\ dy + h\ dz$.*

This is the starting point for the dictionary between the modern approach (one-forms) and the traditional approach (vector fields) to vector calculus.

Example 2.1.4 A one-form and its associated vector field. The expression

$$\omega = x\ dx + x \sin(y)\ dy$$

[1]A subset $U \subseteq \mathbb{R}^2$ is **open** if for every point $P \in U$ there is an open ball (an open disk in $\mathbb{R}^2$, an open sphere in $\mathbb{R}^3$) centered at P that lies entirely in U. This is a generalization of open intervals (and unions of open integrals) to two and three dimensions.

is a one-form on $\mathbb{R}^2$. The associated vector field is the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(x, y) = (x, x\sin(y))$, which assigns the vector $(x, x\sin(y))$ to the point $(x, y) \in \mathbb{R}^2$. □

2.1.3 Exercises

1. Consider the vector field in $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = (x, \sin y, xyz)$. Write the corresponding one-form.

Solution. The corresponding one-form is $\omega = x\ dx + \sin y\ dy + xyz\ dz$.

2. Consider the vector field $\mathbf{v}(x, y) = (-y, x)$ in $\mathbb{R}^2$.

(a) Plot the given vector field at points with integer coordinates.

(b) If the vector field represents the velocity of a flowing fluid, what kind of motion is this vector field describing?

Solution.

(a) We can plot the vector field by plotting arrows for points in $\mathbb{R}^2$ with integer coordinates. For instance, at the point $(1, 1)$, the vector field would be the vector $\mathbf{v}(1, 1) = (-1, 1)$. At $(0, 1)$, we would get $\mathbf{v}(0, 1) = (-1, 0)$. And so on and so forth. The result is the following plot, which was obtained via this geogebra app.

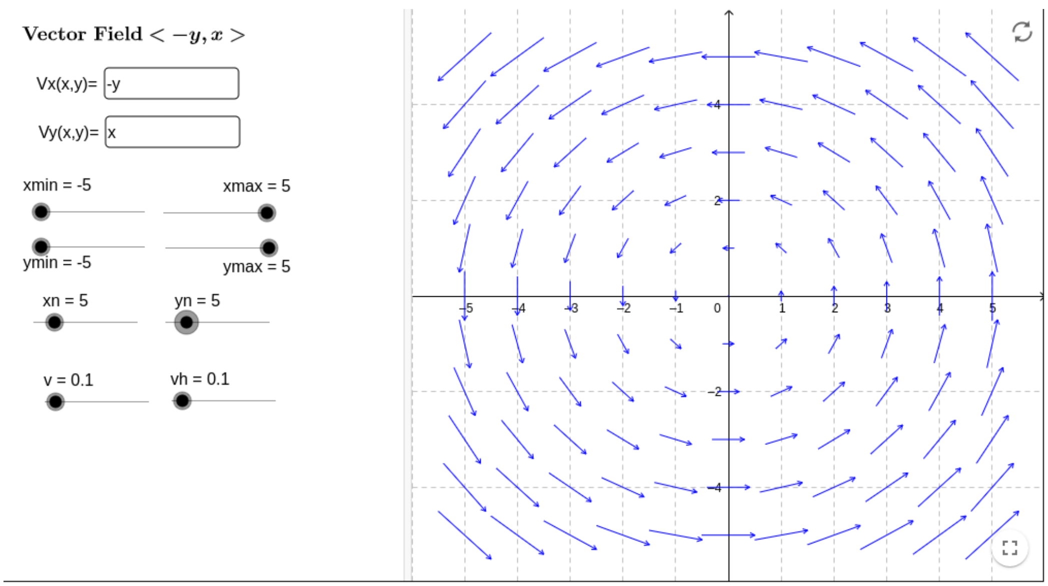


Figure 2.1.5 A plot of the vector field $\mathbf{v}(x, y) = (-y, x)$.

(b) Looking at the plot, we see that the fluid is rotating around the origin counterclockwise. This type of fluid motion is called "vortex".

3. Consider a vector field $\mathbf{v}(x, y) = (5, 1)$. Suppose that it is the velocity field of a moving fluid. Suppose that you drop an object at the origin at time $t = 0$. Where will the object be at time $t = 2$?

Solution. The velocity field $\mathbf{v}(x, y) = (5, 1)$ is constant, which means that all points in the fluid are moving at the same constant velocity. In other words, each point in the fluid is going through a linear motion of the form

$$\mathbf{x}(t) = \mathbf{x}_0 + t(5, 1),$$

where $\mathbf{x}_0$ is some initial position. If we drop an object at the origin at $t = 0$, its position vector is then described by $\mathbf{x}(t) = t(5, 1)$ since its initial position is $\mathbf{x}_0 = (0, 0)$. We conclude that at time $t = 2$, the object is at position $\mathbf{x}(2) = 2(5, 1) = (10, 2)$.

4. Consider the object $\omega = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$. Is this a one-form on $\mathbb{R}^2$?

Solution. ω is not a one-form on $\mathbb{R}^2$, because its component functions are not defined at the origin $(0, 0) \in \mathbb{R}^2$ (we would be dividing by zero). However, it is a one-form on the open subset $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ where we removed the origin,[2] since the component functions are smooth on U.

2.2 Exact one-forms and conservative vector fields

We introduce a class of one-forms that arise as differentials of a function; the associated vector field is the gradient of the function. But not all one-forms are differentials of functions: we call such one-forms "exact", and their corresponding vector fields "conservative".

Objectives

You should be able to:

- Calculate the differential of a function, and intepret it as a one-form.
- Translate between the differential of a function and its gradient.
- Define exact one-forms.
- Define conservative vector fields and their associated potentials.
- Translate between exact one-forms and conservative vector fields.
- Define closed one-forms using partial derivatives.
- Show that exact one-forms are always closed.
- Rephrase the statement as the "screening test" for conservative vector fields.
- Use the screening test to show that a given vector field cannot be conservative.

2.2.1 Differential of a function

One-forms may seem strange, but in fact there is a large class of one-forms that can be obtained directly from functions.

[2] The notation $\mathbb{R}^2 \setminus \{(0, 0)\}$ means "$\mathbb{R}^2$ minus the point $(0, 0)$".

Definition 2.2.1 Differential of a function. Let f be a smooth function on an open subset $U \subseteq \mathbb{R}^3$. Its **differential** (it will become known as "exterior derivative" shortly) is

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

It is a one-form on U. A similar definition of course holds for $\mathbb{R}$, $\mathbb{R}^2$, and in fact $\mathbb{R}^n$. $\lozenge$

Using the correspondence in Principle 2.1.3 between one-forms and vector fields, we can find the vector field associated to the differential of a function.

Fact 2.2.2 Correspondence between the differential and the gradient of a function. *The vector field associated to the differential df of a function f is the vector field with component functions*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

which is nothing else but the gradient $\boldsymbol{\nabla} f$.

Recall from Calculus III that at a given point $(x_0, y_0, z_0) \in U$, the gradient vector field $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ gives the direction of fastest increase of f. In three dimensions, this direction is orthogonal to the level surfaces. In two dimensions, it is perpendicular to the level curves: think of a topographical map, the direction of fastest increase is the direction perpendicular to the contour lines.

Example 2.2.3 The differential and gradient of a function. Consider the function $f(x, y) = x^2 e^y$. Its differential is

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 2xe^y\, dx + x^2 e^y\, dy,$$

which is a one-form on $\mathbb{R}^2$. Its associated vector field is the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\mathbf{F}(x, y) = \left(2xe^y, x^2 e^y\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \boldsymbol{\nabla} f.$$

$\square$

Remark 2.2.4 Using the definition of the differential of a function, we can somewhat make sense of the placeholders dx, dy, and dz. Consider for instance the function $f(x, y, z) = x$. Its differential becomes

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = dx.$$

In other words, the placeholders dx, dy, and dz are simply the differentials of the component functions x, y, and z on $\mathbb{R}^3$.

Note that we will give a more satisfying algebraic meaning for these placeholders later on in Subsection 4.1.1.

2.2.2 Exact one-forms and conservative vector fields

Differentials of functions are one-forms, but not all one-forms are differentials of functions, just like not all vector fields can be written as the gradient of a function. Such one-forms and vector fields are special, and hence have their own name.

Definition 2.2.5 Exact one-forms. We say that a one-form ω on $U \subseteq \mathbb{R}^3$ is **exact** if $\omega = df$ for some function f on U. $\Diamond$

We can define a similar concept for vector fields.

Definition 2.2.6 Conservative vector fields. A vector field $\mathbf{F}$ on $U \subseteq \mathbb{R}^3$ is **conservative** if $\mathbf{F} = \boldsymbol{\nabla} V$ for some function V on U. We call V a **potential** for $\mathbf{F}$.[1] $\Diamond$

Clearly, by Fact 2.2.2, we see that if a one-form is exact, its associated vector field is conservative, and vice-versa.

Example 2.2.7 An exact one-form and its associated conservative vector field. Consider the one-form $\omega = \cos x \; dx - \sin y \; dy$ on $\mathbb{R}^2$. ω is an exact one-form, since it can be written as $\omega = df$ for the function $f(x, y) = \sin x + \cos y$. In the language of vector fields, this is the statement that the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ associated to ω, which is given by $\mathbf{F}(x, y) = (\cos x, -\sin y)$, is conservative, since it can be written as $\mathbf{F} = \boldsymbol{\nabla} f$ for the function f above. The function f is called the potential of the vector field $\mathbf{F}$. $\square$

Example 2.2.8 The gravitational force field is conservative. The gravitational force field $\mathbf{F} : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$ that a mass M at the origin exerts on a mass m at position (x, y, z) is given by

$$\mathbf{F}(x, y, z) = -\frac{GMm}{r^3}(x, y, z),$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and G is the gravitional constant. The corresponding one-form is

$$\omega = -\frac{GMm}{r^3}\left(x \; dx + y \; dy + z \; dz\right).$$

It is easy to see that the gravitational force field is conservative, or equivalently that the one-form ω is exact, since $\omega = d\varphi$ with the function φ given by

$$\varphi(x, y, z) = \frac{GMm}{r}.$$

φ is the potential function, which from physics you may recognize as minus the gravitational potential energy. $\square$

2.2.3 Closed one-forms in $\mathbb{R}^2$

Since not all one-forms can be written as differentials of functions, i.e. not all one-forms are exact, a natural question arises: can we determine, looking at a one-form, whether it is exact or not? Similarly, can we easily determine whether a vector field is conservative? Unfortunately we will not be able to fully answer this question at the moment, we will come back to it in Section 3.6. For the time being, we will be able to find a necessary condition for a one-form to be exact, which in the context of vector calculus is sometimes called a "screening test" for conservative vector fields.

Let us focus first on one-forms and vector fields on $U \subseteq \mathbb{R}^2$.

[1] Note that in physics, the potential of a conservative vector field is usually defined as $\mathbf{F} = -\boldsymbol{\nabla} V$, with an extra minus sign. The difference is purely conventional; the minus sign is introduced so that when $\mathbf{F}$ is a force field, then V becomes the potential energy physically.

Definition 2.2.9 Closed one-forms in $\mathbb{R}^2$. We say that a one-form $\omega = f\ dx + g\ dy$ on $U \subseteq \mathbb{R}^2$ is **closed** if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

◊

This definition may seem *ad hoc*, but it is important because of the following lemma.

Lemma 2.2.10 Exact one-forms in $\mathbb{R}^2$ are closed. *If a one-form ω on $U \subseteq \mathbb{R}^2$ is exact, then it is closed.*

Proof. Suppose that ω is exact: then it can be written as

$$\omega = f\ dx + g\ dy = dF = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

for some smooth function F on $U \subseteq \mathbb{R}^2$. It will then be closed if

$$\frac{\partial f}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

is equal to

$$\frac{\partial g}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

Equality of the two expressions follows from the Clairaut-Schwarz theorem, which states that partial derivatives commute, as long as they are continuous. But continuity of the partial derivatives is guaranteed by the fact that all partial derivatives of F exist and are continuous, since by definition (see Definition 2.1.1) one-forms are assumed to have smooth component functions. Therefore the one-form is closed. ■

Note however that the converse statement is not necessarily true: not all closed one-forms on $U \subseteq \mathbb{R}^2$ are exact. In fact, the question of when closed one-forms are exact is an important one; the result is known as Poincare's lemma. We will come back to this in Section 3.6. But what Lemma 2.2.10 tells us is that one-forms that are not closed cannot be exact.

There is of course an analogous statement for conservative vector fields. The only minor difference is that we need to impose a condition on the component functions of vector fields, since vector fields are not always smooth by definition (see Definition Definition 2.1.2.

Lemma 2.2.11 Screening test for conservative vector fields in $\mathbb{R}^2$. *If a vector field $\mathbf{F} = (f_1, f_2)$ on $U \subseteq \mathbb{R}^2$ is conservative and has component functions that are continuously differentiable, then it passes the **screening test**:*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}.$$

Proof. Same as for Lemma 2.2.10, but for the associated vector fields. ■

In the context of vector fields, it is called a screening test, because it is a quick test to determine whether a vector field has a chance at all to be conservative. In other words, if a vector field does not pass the screening test, then it is certainly not conservative. However, if it passes the screening test, at this stage we cannot conclude anything. Just like closed one-forms are not necessarily exact.

Example 2.2.12 Exact one-forms are closed. Consider the exact one-form on $\mathbb{R}^2$ from Example 2.2.7, $\omega = \cos x \, dx - \sin y \, dy$. We show that it is closed, according to Definition 2.2.9. The partial derivatives are easily calculated:

$$\frac{\partial}{\partial y} \cos x = 0, \qquad \frac{\partial}{\partial x}(-\sin y) = 0.$$

Thus Definition 2.2.9 is satisfied, and ω is closed. □

Example 2.2.13 Closed one-forms are not necessarily exact. If a vector field is conservative, then it passes the screening test. Correspondingly, if a one-form is exact, then it is closed. But the converse statement is not necessarily true (we will revisit it later in Section 3.6). Consider for instance the one-form

$$\omega = -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy.$$

Calculating the partial derivatives for Definition 2.2.9, we get that

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2}, \qquad \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2}.$$

The two expressions are equal, and thus ω is closed. However, one can show that ω is not exact: there does not exist a function f such that $\omega = df$ (see Exercise 3.4.3.2). □

2.2.4 Closed one-forms in $\mathbb{R}^3$

We focused on $\mathbb{R}^2$ for simplicity, but similar results hold for one-forms and vector fields in $\mathbb{R}^3$.

Definition 2.2.14 Closed one-forms in $\mathbb{R}^3$. We say that a one-form $\omega = f \, dx + g \, dy + h \, dz$ on $U \subseteq \mathbb{R}^3$ is **closed** if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

◊

We then have the lemma:

Lemma 2.2.15 Exact one-forms in $\mathbb{R}^3$ are closed. *If a one-form ω on $U \subseteq \mathbb{R}^3$ is exact, then it is closed.*

Proof. Suppose that ω is exact: then it can be written as

$$\omega = f \, dx + g \, dy + h \, dz = dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy + \frac{\partial F}{\partial z} \, dz$$

for some smooth function F on $U \subseteq \mathbb{R}^3$. It will then be closed if

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}, \qquad \frac{\partial^2 F}{\partial z \partial x} = \frac{\partial^2 F}{\partial x \partial z}, \qquad \frac{\partial^2 F}{\partial z \partial y} = \frac{\partial^2 F}{\partial y \partial z}.$$

As before, these equalities follow from the Clairaut-Schwarz theorem, which states that partial derivatives commute as long as they are continuous. ■

The analogous statement for vector fields goes as follows:

Lemma 2.2.16 Screening test for conservative vector fields in $\mathbb{R}^3$. *If a vector field $\mathbf{F} = (f_1, f_2, f_3)$ on $U \subseteq \mathbb{R}^3$ is conservative and has component functions that are continuously differentiable, then it passes the* ***screening test****:*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \qquad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \qquad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}.$$

Proof. Same as for Lemma 2.2.15, but for the associated vector fields. ■

Remark 2.2.17 At this stage the definition of closeness for one-forms and the associated screening test for vector fields appear to be rather *ad hoc*. Sure, they are necessary conditions for a one-form to be exact and a vector field to be conservative, but is that it? No, not really. In fact, those conditions will come out very naturally when we go beyond one-forms and introduce the theory of k-forms in general. We will see how we can "differentiate" forms - this is the notion of exterior derivative. Then the definition of closed one-forms in Definition 2.2.9 will be simply that a one-form ω is closed if its exterior derivative $d\omega$ vanishes. Furthermore, if F is the vector field associated to a one-form ω, then the vector field associated to its derivative $d\omega$ will be called the "curl" of $\mathbf{F}$, and denoted by $\boldsymbol{\nabla} \times \mathbf{F}$. The screening test for vector fields will then be that $\mathbf{F}$ is "curl-free", that is $\boldsymbol{\nabla} \times \mathbf{F} = 0$. All fun stuff, but it will have to wait for a bit! Coming in Chapter 4.

2.2.5 Exercises

1. Consider the function $f(x, y, z) = \sin(y) + zx$. Find its differential df.

Solution. df is given by:

$$\begin{aligned} df =& \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \\ =& z\, dx + \cos(y)\, dy + x\, dz. \end{aligned}$$

2. Consider the one-form $\omega = x\, dx - y\, dy$ on $\mathbb{R}^2$. Show that it is exact, and find a function f such that $\omega = df$.

Solution. Suppose that there exists a function $f(x, y)$ such that $df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = x\, dx - y\, dy$. Then we must have

$$\frac{\partial f}{\partial x} = x, \qquad \frac{\partial f}{\partial y} = -y.$$

Integrating the first equation (recalling that this is a partial derivative, so the "constant of integration" is any function of y alone), we get:

$$f(x, y) = \frac{x^2}{2} + g(y)$$

for some function $g(y)$. Substituting into the second equation, we get

$$\frac{\partial f}{\partial y} = g'(y) = -y, ,$$

which can be integrated to

$$g(y) = -\frac{y^2}{2} + C$$

for any constant C. Since we are only interested in finding one f such that $df = \omega$, we can choose $C = 0$. We conclude that the function

$$f(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$$

is such that $df = \omega$, and hence that ω is an exact one-form.

3. True or False. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative, then $\mathbf{F} + \mathbf{G}$ is also conservative.

Solution. True. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative, then $\mathbf{F} = \boldsymbol{\nabla} f$ and $\mathbf{G} = \boldsymbol{\nabla} g$ for some functions f and g. But then $\mathbf{F} + \mathbf{G} = \boldsymbol{\nabla}(f + g)$, and hence $\mathbf{F} + \mathbf{G}$ is also conservative.

4. Consider the vector field $\mathbf{F}(x, y, z) = (yze^{xy}, xze^{xy}, e^{xy})$. Show that it is conservative and find a potential.

Solution. $\mathbf{F}$ is conservative if there exists a function f such that $\mathbf{F} = \boldsymbol{\nabla} f$. So we need to solve the equations

$$\frac{\partial f}{\partial x} = yze^{xy}, \qquad \frac{\partial f}{\partial y} = xze^{xy}, \qquad \frac{\partial f}{\partial z} = e^{xy}.$$

Let us start by integrating the last one. We get:

$$f(x, y, z) = ze^{xy} + g(x, y)$$

for some function $g(x, y)$. Substituting back into the second one, we get:

$$\frac{\partial f}{\partial y} = xze^{xy} + \frac{\partial g}{\partial y} = xze^{xy},$$

from which we conclude that $\frac{\partial g}{\partial y} = 0$, that is, $g(x, y) = h(x)$ for some function $h(x)$. Finally, substituting back into the first equation, we get:

$$\frac{\partial f}{\partial x} = yze^{xy} + \frac{dh}{dx} = yze^{xy},$$

from which we get $h'(x) = 0$, that is $h(x) = C$ for some constant C. We choose $C = 0$, and we conclude that $\mathbf{F}$ is conservative, with potential

$$f(x, y, z) = ze^{xy}.$$

5. Show that the one-form $\omega = f'(x)\ dx + g'(y)\ dy$ on $\mathbb{R}^2$ is exact, for any smooth functions $f, g : \mathbb{R} \to \mathbb{R}$.

Solution. Consider the function $F(x, y) = f(x) + g(y)$. Its differential is

$$dF = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy = f'(x)\ dx + g'(y)\ dy.$$

Therefore ω is exact, and it is a well defined one-form on $\mathbb{R}^2$ as f and g are smooth.

6. Show that the one-form $\omega = \cos(y)\ dx - x\sin(y)\ dy$ is closed. Is it exact?

Solution. Let us write $\omega = f\ dx + g\ dy$ for $f(x,y) = \cos(y)$ and $g(x,y) = -x\sin(y)$. To show that it is closed, we calculate:

$$\frac{\partial f}{\partial y} = -\sin(y), \qquad \frac{\partial g}{\partial x} = -\sin(y).$$

As the two partial derivatives are equal, we conclude that ω is a closed one-form.

Is it exact? We are looking for a function $F(x,y)$ such that $dF = \omega$, that is,

$$\frac{\partial F}{\partial x} = \cos(y), \qquad \frac{\partial F}{\partial y} = -x\sin(y).$$

Integrating the first equation, we get $F(x,y) = x\cos(y) + h(y)$ for some function $h(y)$. Substituting in the second equation, we get

$$\frac{\partial F}{\partial y} = -x\sin(y) + h'(y) = -x\sin(y),$$

from which we conclude that $h'(y) = 0$, that is $h(y) = C$. We pick $C = 0$. We conclude that $\omega = dF$ with $F(x,y) = x\cos(y)$, and hence it is exact.

7. Show that the vector field $\mathbf{F}(x,y,z) = (y+z, x+z, x+y)$ passes the screening test.

Solution. We simply need to calculate partial derivatives. If we denote the component functions by (f_1, f_2, f_3), we get:

$$\frac{\partial f_1}{\partial y} = 1, \qquad \frac{\partial f_1}{\partial z} = 1, \qquad \frac{\partial f_2}{\partial z} = 1$$
$$\frac{\partial f_2}{\partial x} = 1, \qquad \frac{\partial f_3}{\partial x} = 1, \qquad \frac{\partial f_3}{\partial y} = 1.$$

It follows that $\mathbf{F}$ passes the screening test. In fact, it is easy to show that it is conservative, with a potential given by $f(x,y,z) = xy + xz + yz$.

8. Determine whether the one-form $\omega = x\ dx + x\ dy + z\ dz$ is exact. If it is, find a function f such that $\omega = df$.

Solution. If we write $\omega = f\ dx + g\ dy + h\ dz$, we see right away that

$$\frac{\partial f}{\partial y} = 0,$$

while

$$\frac{\partial g}{\partial x} = 1.$$

Thus ω is not closed, and hence it cannot exact.

9. Consider the vector field $\mathbf{F}(x,y) = (xy^n, 2x^2y^3)$ for some positive integer n. Find the value of n for which $\mathbf{F}$ is conservative on $\mathbb{R}^2$, and find a potential for this value of n.

Solution. For $\mathbf{F}$ to be conservative, it must pass the screening test. We write $\mathbf{F} =$

(f_1, f_2), and calculate:

$$\frac{\partial f_1}{\partial y} = nxy^{n-1}, \qquad \frac{\partial f_2}{\partial x} = 4xy^3.$$

We see that the two partial derivatives are equal for all (x, y) if and only if $n = 4$, in which case the vector field becomes $\mathbf{F} = (xy^4, 2x^2y^3)$. To show that it is conservative and find a potential function, we are looking for a function $f(x, y)$ such that $\boldsymbol{\nabla} f = (xy^4, 2x^2y^3)$. So we must have

$$\frac{\partial f}{\partial x} = xy^4, \qquad \frac{\partial f}{\partial y} = 2x^2y^3.$$

Integrating the first equation, we get

$$f(x, y) = \frac{1}{2}x^2y^4 + g(y)$$

for some function $g(y)$. Substituting in the second equation, we get

$$\frac{\partial f}{\partial y} = 2x^2y^3 + g'(y) = 2x^2y^3,$$

from which we get $g'(y) = 0$, that is $g(y) = C$. We pick $C = 0$. Thus $\mathbf{F}$ is conservative for $n = 4$, and a potential function is given by

$$f(x, y) = \frac{1}{2}x^2y^4.$$

2.3 Changes of variables

Recall from the introduction that our goal is to construct a theory of integration over curves, surfaces, and volumes. Our guiding light is that we want the theory to be "reparametrization-invariant" and "oriented". In the last few subsections, we introduced the concept of one-forms and their associated vector fields. In the next section, we will see that one-forms become the objects that will be integrated over curves. Thus, to study reparametrization-invariance, we need to know how one-forms transform under changes of variables. This is what we study in this section. For the time being, we focuse only on functions and one-forms on $\mathbb{R}$, and remain informal and hand-wavy; our goal is to get a feeling for how one-forms transform. Studying how one-forms transform will encourage us to introduce a little bit of mathematical formalism: the concept of "pullbacks". We will generalize further the concept of pullbacks in the next section.

Objectives

You should be able to:

- Determine how one-forms transform under changes of variables on $\mathbb{R}$.
- State the transformation property of one-forms in terms of the mathematical notion of pullback.

2.3.1 How functions and one-forms on $\mathbb{R}$ transform under changes of variables

Consider a smooth function $f(x)$ of a single variable x, i.e. $f : U \to \mathbb{R}$ for some open subset $U \subseteq \mathbb{R}$. Now suppose that we think of x itself as a smooth function of another variable t, that is, $x = x(t)$. What happens to the function f? Well, it's pretty simple: our function becomes $f(x(t))$, which defines a new function, which we could call $g(t) = f(x(t))$.

What about one-forms? How do these transform under changes of variables? To see what is going on, let us first consider a one-form on $\mathbb{R}$ that is exact, i.e. a differential of a function: $\omega = dF$. We can write the one-form as

$$\omega = dF(x) = \frac{dF}{dx}\ dx$$

in terms of a real variable x. What happens if we think of x itself as a smooth function of another variable t, that is, $x = x(t)$? We can use what we learned above about functions. We are interested in $dF(x(t))$. If we define $G(t) = F(x(t))$ as above, then our one-form can be written as $dG(t) = \frac{dG}{dt}\ dt$. But, using the chain rule of calculus, we know that

$$\frac{dG}{dt} = \frac{d}{dt}\left(F(x(t)\right) = \frac{dF}{dx}\frac{dx}{dt}.$$

So by changing variable from x to t, we get a new one-form, let's call it η, defined by

$$\eta = \frac{dG}{dt}\ dt = \left(\frac{dF}{dx}\frac{dx}{dt}\right)\ dt.$$

We now generalize this to all one-forms on $\mathbb{R}$ (or open subsets thereof), not just exact one-forms. Given a one-form $\omega = f(x)\ dx$ written in terms of a variable x, if we think of $x = x(t)$ as a function of a new variable t, then by changing variable from x to t we get a new one-form η:

$$\eta = \left(f(x(t))\frac{dx}{dt}\right)\ dt.$$

This defines how one-forms transform under changes of variables. As we will see, this transformation property is what lies behind the substitution formula for definite integrals.

Remark 2.3.1 The upshot of this brief discussion is that it is easy to remember how one-forms in $\mathbb{R}$ transform under changes of variables. If we write $\omega = f(x)\ dx$, and do a change of variable $x = x(t)$, then all we need to do is rewrite the coefficient function as a function of t by composition $f(x(t))$, and then "transform the differential" dx as $dx(t) = \frac{dx}{dt}\ dt$. This gives us the new one-form $\eta = f(x(t))\frac{dx}{dt}\ dt$.

Example 2.3.2 An example of a change of variables. Consider the one-form $\omega = x^2 dx = f(x)\ dx$ on $\mathbb{R}$. Let us do the change of variables $x = \sin t$, with $\frac{dx}{dt} = \cos t$. By changing variables from x to t we a get a new one-form

$$\eta = f(x(t))\frac{dx}{dt}\ dt = (\sin t)^2 \cos t\ dt.$$

$\square$

2.3.2 The pullback of functions and one-forms on $\mathbb{R}$

Our brief discussion above can be formalized mathematically in terms of the concept of "pullback". Let us start with functions again, and be a little more formal. Given a function $f : U \to \mathbb{R}$ for some open subset $U \subseteq \mathbb{R}$. We write $f(x)$ in terms of a variable x. Now suppose that $x = \phi(t)$, for some smooth function $\phi : V \to U$, where $V \subseteq \mathbb{R}$ is also an open subset of $\mathbb{R}$. As described above, changing variables from x to t amounts to defining a new function $g(t) = f(\phi(t))$. This new function is simply the composition of f and ϕ:

$$g = f \circ \phi : V \to \mathbb{R}.$$

We call this new function "the pullback of f".

Definition 2.3.3 The pullback of a function on $\mathbb{R}$. Let $U \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ be open subsets, and $f : U \to \mathbb{R}$ and $\phi : V \to U$ be smooth functions. The **pullback of** f, which is denoted by $\phi^* f$, is the smooth function

$$\phi^* f := f \circ \phi : V \to \mathbb{R}.$$

Explicitly, the pullback can be written as $\phi^* f(t) = f(\phi(t))$. ◊

In other words, the pullback of a function by another function just means that we are composing functions. It is called "pullback" because if we think of the chain of maps: $V \xrightarrow{\phi} U \xrightarrow{f} \mathbb{R}$, while our original function was from U to $\mathbb{R}$, by composition we "pull it back" to a function from V to $\mathbb{R}$.

We can define a similar concept for one-forms on $\mathbb{R}$, using the discussion above about how they transform under changes of variables.

Definition 2.3.4 The pullback of a one-form on $\mathbb{R}$. Let $U \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ be open subsets, $\omega = f(x)\, dx$ be a one-form on U, and $\phi : V \to U$ be a smooth function. The **pullback of** ω, which is denoted by $\phi^*\omega$, is the one-form on V defined by

$$\phi^*\omega = \left(f(\phi(t)) \frac{d\phi}{dt} \right)\, dt = \left(\phi^* f(t) \frac{d\phi}{dt} \right)\, dt$$

◊

Note that when calculating the pullback of a one-form, it is very important not to forget the $\frac{d\phi}{dt}$ term!

Example 2.3.5 Change of variables as pullback. Going back to Example 2.3.2, we could rephrase it as follows. We have a one-form $\omega = x^2 dx = f(x)\, dx$ on $\mathbb{R}$, and a function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = \sin t$. The pullback one-form $\phi^*\omega$ is then given by

$$\phi^*\omega = f(\phi(t)) \frac{d\phi}{dt}\, dt = (\sin t)^2 \cos t\, dt.$$

This is of course the same thing as implementing the change of variables $x \to t$ in the one-form ω. □

2.3.3 Exercises

1. Consider the one-form $\omega = e^x \sin(x)\ dx$ on $\mathbb{R}$, and the smooth function $\phi : \mathbb{R}_{>0} \to \mathbb{R}$ with $\phi(t) = \ln(t)$ (where $\mathbb{R}_{>0}$ is the set of positive real numbers). Find the pullback one-form $\phi^*\omega$. Where is the one-form $\phi^*\omega$ defined?

Solution. First, since $\phi : \mathbb{R}_{>0} \to \mathbb{R}$, and ω is a one-form defined on all of $\mathbb{R}$, by definition of the pullback we see that the pullback one-form $\phi^*\omega$ is defined only on $\mathbb{R}_{>0}$, i.e. for all positive real numbers. Using the definition of pullback, we find its expression as:

$$\begin{aligned}\phi^*\omega =& e^{\phi(t)} \sin(\phi(t)) \frac{d\phi}{dt}\ dt\\ =& e^{\ln(t)} \sin(\ln(t)) \frac{1}{t}\ dt\\ =& t \sin(\ln(t)) \frac{1}{t}\ dt\\ =& \sin(\ln(t))\ dt.\end{aligned}$$

2. Consider the one-form $\omega = \frac{1}{x}\ dx$ defined on $\mathbb{R}_{>0}$, and the smooth function $\phi : \mathbb{R} \to \mathbb{R}_{>0}$ defined by $\phi(t) = e^t$. Find the pullback one-form $\phi^*\omega$.

Solution. First, by definition of the pullback we see that $\phi^*\omega$ is defined on all of $\mathbb{R}$. We find its expression to be:

$$\begin{aligned}\phi^*\omega =& \frac{1}{\phi(t)} \phi'(t)\ dt\\ =& e^{-t} e^t\ dt\\ =& dt.\end{aligned}$$

How simple! :-)

This is not really a surprise, since $\omega = d\ln(x)$. One property of the pullback of one-forms is that for exact one-forms, $\phi^* df = d(\phi^* f)$. For $f = \ln(x)$, the pullback of the function is simply $\phi^* f(t) = f(\phi(t)) = \ln(e^t) = t$, and hence $\phi^* df = d\phi^* f = dt$, as we found.

3. Following up on the previous exercise, show that for an exact one-form $\omega = df$ on $U \subseteq \mathbb{R}$, and a smooth function $\phi : V \to U$, the following property of the pullback is satisfied:

$$\phi^*(df) = d(\phi^* f).$$

In other words, the pullback commutes with the exterior derivative of a function.

Solution. From the definition of the pullback of a function, we can write the right-hand-side as:

$$d(\phi^* f) = d(f(\phi(t)) = \frac{df(\phi(t))}{dt}\ dt.$$

Using the chain rule, this can be written as

$$\begin{aligned}d(\phi^* f) =& f'(\phi(t)) \frac{d\phi}{dt}\ dt\\ =& \phi^*(f'(x)\ dx)\end{aligned}$$

$$=\phi^*(df),$$

where we used the definition of the pullback of a one-form. This concludes the proof.

2.4 The pullback of a one-form

All right, time to get serious! :-) In the previous section we introduced the notion of the pullback of a one-form with respect to a function $\phi : V \to U$ with $U, V \subseteq \mathbb{R}$ open subsets. But this notion of pullback can be generalized, and will become essential to develop our theory of integration (in fact, perhaps this class should be called "the power of the pullback"!). In this section we provide a more general definition of the pullback of a one-form.

Objectives

You should be able to:

- Determine the pullback of a one-form in general.
- State and use the three fundamental properties of the pullback of one-forms.

Let ω be a one-form on $U \subseteq \mathbb{R}^n$, where $n \in \{1, 2, 3\}$, i.e. we can be in one, two, or three dimensions. We now consider a smooth function $\phi : V \to U$, where $V \subseteq \mathbb{R}^m$, with again $m \in \{1, 2, 3\}$. Note that m and n don't have to be the same: we could have, say $U \subseteq \mathbb{R}^3$, and $V \subseteq \mathbb{R}^2$. Our goal is to define the pullback $\phi^*\omega$, which should be a one-form on V.

Just to be concrete: we could take, for instance, a one-form ω on $\mathbb{R}^3$, and a smooth function $\phi : \mathbb{R}^2 \to \mathbb{R}^3$. The pullback $\phi^*\omega$ should then be a one-form on $\mathbb{R}^2$.

Note that our notion of pullback should generalize the definition of pullback in Definition 2.3.4, which should consist in the case with $m = n = 1$.

2.4.1 The pullback of a function

Let us first define the pullback of a function in this context, generalizing Definition 2.3.3.

Definition 2.4.1 The pullback of a function. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, where $m, n \in \{1, 2, 3\}$. Let $f : U \to \mathbb{R}$ and $\phi : V \to U$ be smooth functions. The **pullback of** f, which is denoted by $\phi^* f$, is the smooth function

$$\phi^* f = f \circ \phi : V \to \mathbb{R}.$$

Explicitely, if we write $\mathbf{t} \in V$ for an m-dimensional vector in V, then

$$\phi^* f(\mathbf{t}) = f(\phi(\mathbf{t})),$$

where $\phi(\mathbf{t})$ is an n-dimensional vector in U. $\Diamond$

Example 2.4.2 The pullback of a function from $\mathbb{R}^3$ to $\mathbb{R}$. To make things more concrete, let us look at a specific example. Suppose that f is a smooth function on $\mathbb{R}^3$, that is $f : \mathbb{R}^3 \to \mathbb{R}$. Let $\phi : \mathbb{R} \to \mathbb{R}^3$ be another smooth function, which takes a point in $\mathbb{R}$ and maps it to a vector in $\mathbb{R}^3$ (so it is a vector-valued function). We can write f explicitly as

$f = f(x, y, z)$. As for the vector-valued function ϕ, we write $\phi(t) = (x(t), y(t), z(t))$. Then the pullback $\phi^* f : \mathbb{R} \to \mathbb{R}$ is simply the composition:

$$\phi^* f(t) = f(\phi(t)) = f(x(t), y(t), z(t)).$$

For instance, if $f(x, y, z) = xy + z$, and $\phi(t) = (t, t^2, 1)$, then

$$\phi^* f(t) = f(t, t^2, 1) = t^3 + 1.$$

□

Example 2.4.3 The pullback of a function from $\mathbb{R}^3$ to $\mathbb{R}^2$. We can do the same thing but pulling back to $\mathbb{R}^2$ instead of $\mathbb{R}$. Suppose that f is a smooth function on $\mathbb{R}^3$, that is $f : \mathbb{R}^3 \to \mathbb{R}$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be another smooth function, which takes a point in $\mathbb{R}^2$ and maps it to a vector in $\mathbb{R}^3$. We can write f explicitly as $f = f(x, y, z)$. As for the vector-valued function ϕ, we write $\phi(t_1, t_2) = (x(t_1, t_2), y(t_1, t_2), z(t_1, t_2))$. Then the pullback $\phi^* f : \mathbb{R} \to \mathbb{R}$ is simply the composition:

$$\phi^* f(t_1 t_2) = f(\phi(t_1, t_2)) = f(x(t_1, t_2), y(t_1, t_2), z(t_1, t_2)).$$

For instance, if $f(x, y, z) = xy + z$, and $\phi(t_1, t_2) = (t_1, t_2^2, t_1 + t_2)$, then

$$\phi^* f(t_1, t_2) = f(t_1, t_2^2, t_1 + t_2) = t_1 t_2^2 + t_1 + t_2.$$

□

2.4.2 An axiomatic definition of the pullback of a one-form

We will take an axiomatic approach to the definition of the pullback of a one-form. Let us first recall three important properties of one-forms (from Subsection 2.1.1 and Definition 2.2.5):

1. If ω and η are one-forms on U, then $\omega + \eta$ is a one-form on U.
2. If ω is a one-form on U and f a smooth function on U, then $f\omega$ is a one-form on U.
3. An exact one-form is a one-form ω that can be written as the differential of a function f on U: $\omega = df$.

We now want the pullback to be consistent with these properties. More precisely, we require that the pullback ϕ^* satisfies the following properties:

1. $\phi^*(\omega + \eta) = \phi^*\omega + \phi^*\eta$.
2. $\phi^*(f\omega) = (\phi^* f)(\phi^*\omega)$.
3. $\phi^*(df) = d(\phi^* f)$.

It turns out that this is completely sufficient to fully determine the pullback of any one-form. Let us see why.

Lemma 2.4.4 The pullback of dx. *Let dx be the basic one-form on $\mathbb{R}^3$. Let $\phi : V \to \mathbb{R}^3$ be a smooth function, where $V \subseteq \mathbb{R}^m$ is an open subset, with $m \in \{1, 2, 3\}$. Write $\mathbf{t} = (t_1, \ldots, t_m)$*

for an m-dimensional vector in V, and $\phi(\mathbf{t}) = (x(\mathbf{t}), y(\mathbf{t}), z(\mathbf{t}))$. Then

$$\phi^*(dx) = \sum_{i=1}^{m} \frac{\partial x}{\partial t_i} dt_i.$$

As an example, if $m = 3$, we would get

$$\phi^*(dx) = \frac{\partial x}{\partial t_1} dt_1 + \frac{\partial x}{\partial t_2} dt_2 + \frac{\partial x}{\partial t_3}\ dt_3.$$

The same formula for $\phi^*(dx)$ remains true if dx is considered to be a one-form on $\mathbb{R}^2$ or $\mathbb{R}$ instead of $\mathbb{R}^3$.

We also note that the same result holds for the basic one-forms dy and dz, with x replaced by y and z respectively.

Proof. This follows from the third axiomatic property that we are imposing on the pullback. Recall from Remark 2.2.4 that we can think of dx as the differential df of the function $f(x, y, z) = x$. By the third property of pullbacks, we want to impose that

$$\phi^*(dx) = \phi^*(df) = d(\phi^* f).$$

From Definition 2.4.1, we can calculate $\phi^* f$. We get $\phi^* f(\mathbf{t}) = x(\mathbf{t})$. We thus obtain

$$\phi^*(dx) = dx(\mathbf{t}) = \sum_{i=1}^{m} \frac{\partial x}{\partial t_i} dt_i,$$

where we use the definition of the differential of the function $x(\mathbf{t})$. ■

This result enables us to write down a general formula for the pullback of a one-form. For clarity, we will only write it down for a one-form on $\mathbb{R}^3$, but it is clear what the similar result should be for a one-form in $\mathbb{R}^2$ or $\mathbb{R}$.

Lemma 2.4.5 The pullback of a one-form. *Let $\omega = f dx + g dy + h dz$ be a one-form on an open subset $U \subseteq \mathbb{R}^3$. Let $\phi : V \to U$ be a smooth function, where $V \subseteq \mathbb{R}^m$ is an open subset, with $m \in \{1, 2, 3\}$. Write $\mathbf{t} = (t_1, \ldots, t_m)$ for an m-dimensional vector in V, and $\phi(\mathbf{t}) = (x(\mathbf{t}), y(\mathbf{t}), z(\mathbf{t}))$. Then*

$$\phi^*\omega = f(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial x}{\partial t_i} dt_i + g(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial y}{\partial t_i} dt_i + h(\phi(\mathbf{t})) \sum_{i=1}^{m} \frac{\partial z}{\partial t_i} dt_i.$$

Proof. To prove this result, we use Lemma 2.4.4 (and the similar result for dy and dz), and the first and second axiomatic properties that we are imposing on the pullback. Using the first and second properties, we can write:

$$\phi^*\omega = \phi^*(f dx + g dy + h dz) = (\phi^* f)\phi^*(dx) + (\phi^* g)\phi^*(dy) + (\phi^* h)\phi^*(dz).$$

We then use Lemma 2.4.4 to evaluate $\phi^*(dx), \phi^*(dy)$ and $\phi^*(dz)$, and from Definition 2.4.1 we know that $\phi^* f(\mathbf{t}) = f(\phi(\mathbf{t}))$, and similarly for g and h. ■

Example 2.4.6 The pullback of a one-form from $\mathbb{R}^3$ to $\mathbb{R}$. Suppose that $\omega = f dx + g dy + h dz$ is a one-form on $\mathbb{R}^3$. Let $\phi : \mathbb{R} \to \mathbb{R}^3$ be a smooth function, which takes a point in $\mathbb{R}$ and maps it to a vector in $\mathbb{R}^3$. We write $\phi(t) = (x(t), y(t), z(t))$. Then the pullback $\phi^*\omega$ is a one-form on $\mathbb{R}$ given by:

$$\phi^*\omega = \left(f(\phi(t))\frac{dx}{dt} + g(\phi(t)\frac{dy}{dt} + h(\phi(t))\frac{dz}{dt} \right) \ dt.$$

For instance, if $\omega = xdx + xydy + z^2dz$, and $\phi(t) = (x(t), y(t), z(t)) = (t^2, t, 1)$, then

$$\begin{aligned}\phi^*\omega =& \left(x(t)\frac{dx}{dt} + x(t)y(t)\frac{dy}{dt} + z(t)^2\frac{dz}{dt} \right) dt\\ =& \Big((t^2)(2t) + (t^2)(t)(1) + (1)(0) \Big) dt\\ =& 3t^3 dt.\end{aligned}$$

□

Example 2.4.7 The pullback of a one-form from $\mathbb{R}^3$ to $\mathbb{R}^2$. Suppose that $\omega = f dx + g dy + h dz$ is a one-form on $\mathbb{R}^3$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth function, which takes a point in $\mathbb{R}^2$ and maps it to a vector in $\mathbb{R}^3$. We write $\phi(\mathbf{t}) = (x(\mathbf{t}), y(\mathbf{t}), z(\mathbf{t}))$, with $\mathbf{t} = (t_1, t_2)$. Then the pullback $\phi^*\omega$ is a one-form on $\mathbb{R}^2$ given by:

$$\begin{aligned}\phi^*\omega =& f(\phi(\mathbf{t}))\left(\frac{\partial x}{\partial t_1}dt_1 + \frac{\partial x}{\partial t_2}dt_2 \right) + g(\phi(\mathbf{t}))\left(\frac{\partial y}{\partial t_1}dt_1 + \frac{\partial y}{\partial t_2}dt_2 \right)\\ &+ h(\phi(\mathbf{t}))\left(\frac{\partial z}{\partial t_1}dt_1 + \frac{\partial z}{\partial t_2}dt_2 \right).\end{aligned}$$

For instance, if $\omega = xdx + xydy + z^2dz$, and $\phi(\mathbf{t}) = (x(\mathbf{t}), y(\mathbf{t}), z(\mathbf{t})) = (t_1t_2, t_2, t_1 + t_2)$, then

$$\begin{aligned}\phi^*\omega =& x(\phi(\mathbf{t}))\left(\frac{\partial x}{\partial t_1}dt_1 + \frac{\partial x}{\partial t_2}dt_2 \right) + x(\phi(\mathbf{t}))y(\phi(\mathbf{t}))\left(\frac{\partial y}{\partial t_1}dt_1 + \frac{\partial y}{\partial t_2}dt_2 \right)\\ &+ z(\phi(\mathbf{t}))^2\left(\frac{\partial z}{\partial t_1}dt_1 + \frac{\partial z}{\partial t_2}dt_2 \right).\\ =& (t_1t_2)(t_2dt_1 + t_1dt_2) + (t_1t_2)(t_2)(dt_2) + (t_1 + t_2)^2(dt_1 + dt_2)\\ =& (t_1t_2^2 + (t_1 + t_2)^2)dt_1 + (t_1^2t_2 + t_1t_2^2 + (t_1 + t_2)^2)dt_2.\end{aligned}$$

□

Example 2.4.8 Consistency check: the pullback of a one-form from $\mathbb{R}$ to $\mathbb{R}$. As a consistency check, we show that the pullback of a one-from from $\mathbb{R}$ to $\mathbb{R}$ reduces to Definition 2.3.4. Let $\omega = f(x)dx$ on $U \subseteq \mathbb{R}$, and $\phi : V \to U$ with $V \subseteq \mathbb{R}$. We write $\phi(t) = x(t)$. Then the pullback $\phi^*\omega$ is the one-form on V given by:

$$\phi^*\omega = \left(f(x(t))\frac{dx}{dt} \right) dt,$$

which indeeds reproduces Definition 2.3.4 with our notation $\phi(t) = x(t)$. □

We now have a very general definition of the pullback of a one-form. This will turn out to be very useful to define the integral of a one-form, which is what we now turn to.

2.4.3 Exercises

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = e^{x+y} + x + y$, and the function $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\phi(u, v, w) = (u + v, v + w)$. Find the pullback $\phi^* f$. What is its domain?

Solution. First, as $f : \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}^2$, we see that the composition $\phi^* f = f \circ \phi : \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$, i.e. the pullback $\phi^* f$ is a function from $\mathbb{R}^3$ to $\mathbb{R}$. So its domain is $\mathbb{R}^3$.

We calculate its expression by composition:

$$\begin{aligned}\phi^* f(u, v, w) =& f(u + v, v + w)\\ =& e^{(u+v)+(v+w)} + (u + v) + (v + w)\\ =& e^{u+2v+w} + u + 2v + w.\end{aligned}$$

2. Consider the one-form $\omega = x^2\ dx$ on $\mathbb{R}$, and the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ given by $\phi(u, v) = u$, which projects on the u-axis. Find the pullback one-form $\phi^*\omega$ on $\mathbb{R}^2$. Interpret the result.

Solution. Let us write $\omega = f\ dx = x^2\ dx$. By the definition of pullback, we get:

$$\begin{aligned}\phi^*\omega =& f(\phi(u, v)) \left(\frac{\partial \phi}{\partial u}\ du + \frac{\partial \phi}{\partial v}\ dv\right)\\ =& u^2\ (1\ du + 0\ dv)\\ =& u^2\ du.\end{aligned}$$

We see that the pullback one-form looks the same, but written in terms of u instead of x. However, $\phi^*\omega$ is defined on $\mathbb{R}^2$, while ω was defined on $\mathbb{R}$. Since the function ϕ here simply projects on the u-axis, what the pullback does here is extend the one-form uniformly in the v-coordinate on the uv-plane; at any two points (u, v_1) and (u, v_2), the one-form will be the same. Conceptually, this is what happens when we pullback using a "forgetful map", i.e. a map that somehow "forgets" some information (in this case, the v-coordinate). The pullback then extends the object uniformly across the forgotten structure.

3. Consider the one-form $\omega = x^2\ dx + y^2\ dy$ on $\mathbb{R}^2$, and the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Phi(r, \theta) = (r \cos\theta, r \sin\theta)$, which defines polar coordinates. Find the pullback $\Phi^*\omega$.

Solution. We write $\omega = f\ dx + g\ dy = x^2\ dx + y^2\ dy$, and $\Phi(r, \theta) = (x(r, \theta), y(r, \theta))$. Then:

$$\begin{aligned}\Phi^*\omega =& f(\Phi(r, \theta)) \left(\frac{\partial x}{\partial r}\ dr + \frac{\partial x}{\partial \theta}\ d\theta\right) + g(\Phi(r, \theta)) \left(\frac{\partial y}{\partial r}\ dr + \frac{\partial y}{\partial \theta}\ d\theta\right)\\ =& r^2 \cos^2\theta\,(\cos\theta\ dr - r \sin\theta\ d\theta) + r^2 \sin^2\theta\,(\sin\theta\ dr + r \cos\theta\ d\theta)\\ =& r^2(\cos^3\theta + \sin^3\theta)\ dr + r^3(\sin^2\theta \cos\theta - \cos^2\theta \sin\theta)\ d\theta.\end{aligned}$$

The notion of pullback allows us to easily calculate how one-forms change under changes of coordinates, such as going from Cartesian to polar coordinates in this case.

4. Consider the one-form $\omega = \frac{z^2\ dz}{\sqrt{x^2+y^2}}$ on $U = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) \neq (0,0,z)\}$ (this is $\mathbb{R}^3$ with the z-axis removed), and the function $\phi : V \to U$ with $\phi(r,\theta,\zeta) = r(\cos\theta, \sin\theta, \zeta)$, and $V = \{(r,\theta,\zeta) \in \mathbb{R}^3 \mid r \neq 0\}$. Determine the pullback one-form $\phi^*\omega$.

Solution. By definition of the pullback, we get:

$$\begin{aligned}\phi^*\omega =& \frac{1}{r} r^2\zeta^2 \left(\frac{\partial}{\partial r}(r\zeta)\ dr + \frac{\partial}{\partial \theta}(r\zeta)\ d\theta + \frac{\partial}{\partial \zeta}(r\zeta) d\zeta \right) \\ =& r\zeta^2(\zeta\ dr + r\ d\zeta).\end{aligned}$$

5. Let ω be a one-form on $\mathbb{R}^3$, and $Id : \mathbb{R}^3 \to \mathbb{R}^3$ the identity function defined by $Id(x,y,z) = (x,y,z)$. Show that $Id^*\omega = \omega$.

Solution. Write $\omega = f\ dx + g\ dy + h\ dz$. By definition of the pullback, we get:

$$\begin{aligned}Id^*\omega =& f(x,y,z) \left(\frac{\partial}{\partial x}(x)\ dx + \frac{\partial}{\partial y}(x)\ dy + \frac{\partial}{\partial z}(x)\ dz \right) \\ &+ g(x,y,z) \left(\frac{\partial}{\partial x}(y)\ dx + \frac{\partial}{\partial y}(y)\ dy + \frac{\partial}{\partial z}(y)\ dz \right) \\ &+ h(x,y,z) \left(\frac{\partial}{\partial x}(z)\ dx + \frac{\partial}{\partial y}(z)\ dy + \frac{\partial}{\partial z}(z)\ dz \right) \\ =& f(x,y,z)\ dx + g(x,y,z)\ dy + h(x,y,z)\ dz \\ =& \omega,\end{aligned}$$

which completes the proof.

6. Let ω be a one-form $U \subseteq \mathbb{R}$, and $\phi : V \to U$ and $\alpha : W \to V$ be smooth functions, with $V, W \subseteq \mathbb{R}$ open subsets. Show that

$$(\phi \circ \alpha)^*\omega = \alpha^*(\phi^*\omega).$$

In other words, it doesn't matter whether we pullback in one or two steps through the chain of maps $W \stackrel{\alpha}{\to} V \stackrel{\phi}{\to} U$.

We note here that while the exercise is only asking you to prove it for open subsets $U, V, W \subseteq \mathbb{R}$, this property is true in general, not just in $\mathbb{R}$.

Solution. Let us write $\omega = f(x)\ dx$, $\phi = \phi(t)$, and $\alpha = \alpha(u)$. On the one hand, we have:

$$(\phi \circ \alpha)^*\omega = f(\phi(\alpha(u))) \frac{d}{du}(\phi(\alpha(u)))\ du.$$

On the other hand, we have

$$\phi^*\omega = f(\phi(t)) \frac{d}{dt}(\phi(t))\ dt,$$

and

$$\alpha^*(\phi^*\omega) = f(\phi(\alpha(u)) \left(\frac{d}{dt}\phi(t) \right) \Big|_{t=\alpha(u)} \frac{d}{du}\alpha(u)\ du.$$

But

$$\frac{d}{du}(\phi(\alpha(u))) = \left(\frac{d}{dt}\phi(t) \right) \Big|_{t=\alpha(u)} \frac{d}{du}\alpha(u)$$

by the chain rule, and hence $(\phi \circ \alpha)^*\omega = \alpha^*(\phi^*\omega)$.

Chapter 3

Integrating one-forms: line integrals

We study how one-forms can be integrated along curves, which leads to the definition of (oriented) line integrals (also called "work integrals"). An important result in this section is the Fundamental Theorem of line integrals, which is the natural generalization of the Fundamental Theorem of Calculus to integrals of one-forms along curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.1 Integrating a one-form over an interval in $\mathbb{R}$

Our goal is to define the integral of a one-form along a curve. But as a starting point, we define how to integrate one-forms over a closed interval in $\mathbb{R}$. We attach particular importance to our guiding principles of reparametrization-invariance and orientability.

Objectives

You should be able to:

- Define the integral of a one-form over an interval in $\mathbb{R}$.
- Determine how the integral changes when reversing the orientation of the interval.
- Relate the transformation property of one-forms to the substitution formula for definte integrals.

3.1.1 The integral of a one-form over an interval

Consider a one-form $\omega = f(x)\ dx$ over an open subset $U \subseteq \mathbb{R}$ that contains the interval $[a, b]$. We define the integral of ω over $[a, b]$.

Definition 3.1.1 The integral of a one-form over $[a, b]$. We define **the integral of ω over $[a, b]$**, with $a \leq b$, as follows:

$$\int_{[a,b]} \omega = \int_a^b f(x)\ dx,$$

where on the right-hand-side we use the standard definition of definite integrals from calculus.

$\Diamond$

Well, that was simple! This is just the standard notion of definite integrals from calculus. All we are doing is introducing some fancy notation for it. Great!

Example 3.1.2 An example of an integral of a one-form over an interval. Consider the one-form $\omega = x^3 \ dx$ on $\mathbb{R}$. Suppose that you want to integrate it over the interval $[0,1]$. Then the integral is

$$\int_{[0,1]} \omega = \int_0^1 x^3 \ dx = \frac{1}{4},$$

where we used the Fundamental Theorem of Calculus to evaluate the integral as usual, since we are back in the realm of the definite integrals that we know and love. □

What is interesting however is to study what our guiding principles of reparametrization-invariance and orientability become in this simple context. Let us start with orientability.

3.1.2 Integrals of one-forms over intervals are oriented

If we look at Definition 3.1.1, there is something a bit peculiar. On the left-hand-side, we are integrating over an interval $[a,b]$, so by definition we must have $a \leq b$. However, on the right-hand-side, we could exchange the limits of integration, and instead of integrating from a to b, we could integrate from b to a. But what would that correspond to on the left-hand-side?

What is going on here is that the definition involves an implicit choice of orientation for the interval. Indeed, when we write $\int_a^b f(x) \ dx$, we say that we integrate "from a to b": this is a choice of direction, of orientation. We think of the interval $[a,b]$ as being implicitly given a choice of direction of increasing real numbers, i.e. from a to b.

But we could have decided to consider the interval $[a,b]$, but with the opposite choice of orientation, i.e. going from b to a, in the direction of decreasing real numbers. That would be another choice of orientation for the interval. Let us be a little more precise.

Definition 3.1.3 The orientation of an interval. We define the **orientation of an interval in** $\mathbb{R}$ to be a choice of direction. There are two choices: either in the direction of increasing real numbers, or decreasing real numbers.

Let $a \leq b$. By $[a,b]_+$, we mean the interval $[a,b]$ with the orientation of increasing real numbers, i.e. "from a to b". By $[a,b]_-$, we denote the same interval but with the orientation of decreasing real numbers (from b to a).

We define the **canonical orientation** to be the orientation of increasing real numbers. When we write $[a,b]$ without specifying the orientation, we always mean the interval with its canonical orientation. ◊

With this clarification, we can extend Definition 3.1.1 to intervals with the opposite choice of orientation.

Definition 3.1.4 The oriented integral of a one-form. We define **the integral of** ω **over the oriented interval** $[a,b]_\pm$, with $a \leq b$, as follows:

$$\int_{[a,b]_\pm} \omega = \pm \int_a^b f(x) \ dx.$$

◊

When the orientation is canonical, that is $[a,b]_+ = [a,b]$, we recover our original definition Definition 3.1.1. But when the interval is oriented in the direction of decreasing real numbers, that is $[a,b]_-$, we get:

$$\int_{[a,b]_-} \omega = -\int_a^b f(x)\ dx = \int_b^a f(x)\ dx,$$

where we used properties of definite integrals to exchange the limits of integration. In other words, we can think of the integral on the right-hand-side as "going from b to a", which is consistent with the orientation of decreasing real numbers as $a \leq b$.

3.1.3 Orientation-preserving reparametrizations

So we know that integrals of one-forms over intervals are oriented. What about reparametrization-invariance?

Suppose that $\omega = f(x)\ dx$ is a one-form over an open subset $U \subseteq \mathbb{R}$ that contains the interval $[a,b]$. What happens to the integral if we do a change of variables $x = \phi(t)$?

We know how to study this question, as we know how one-forms transform under changes of variables. The key: the pullback. Let us be a bit more precise.

Suppose that we are given a function $\phi : [c,d] \to [a,b]$, and assume that ϕ can be extended to a C^1-function[1] on an open set $V \subseteq \mathbb{R}$ that contains $[c,d]$ (we call V and "open neighborhood of $[c,d]$"). We know how the one-form changes under this change of variables: as in Section 2.3, we can pullback ω to get a new one-form $\phi^*\omega$ which can be integrated over $[c,d]$. The question is: does the integral change when we do such a change of variables? In other words, is the integral of $\phi^*\omega$ over $[c,d]$ equal to the integral of ω over $[a,b]$?

To answer this question we need to be a little more precise. We impose a further requirement on ϕ: we require that $\phi(c) = a$ and $\phi(d) = b$, that is, it maps the left endpoint of the interval $[c,d]$ to the left endpoint of the interval $[a,b]$, and similarly for the right endpoints. This means that ϕ "preserves the orientation" of the intervals: it maps the smallest real number to the smallest one, and the largest one to the largest one. In other words, it preserves the direction of increasing real numbers.

Note that we can think of ϕ as a "reparametrization", in the sense that we can think of the interval $[a,b] \subset \mathbb{R}$ as a "curve" in $\mathbb{R}$, and ϕ as a parametrization of the curve.

Lemma 3.1.5 Integrals of one-forms over intervals are invariant under orientation-preserving reparametrizations. *Let ω be a one-form on $U \subseteq \mathbb{R}$ with $[a,b] \subset U$, and $\phi : [c,d] \to [a,b]$ be a function that can be extended to a C^1-function on an open neighborhood of $[c,d]$ and such that $\phi(c) = a$ and $\phi(d) = b$. Then*

$$\int_{[c,d]} \phi^*\omega = \int_{[a,b]} \omega.$$

Explicitly, $\omega = f(x)\ dx$, and $\phi^*\omega = f(\phi(t))\frac{d\phi}{dt}\ dt$, thus the statement becomes

$$\int_c^d f(\phi(t))\frac{d\phi}{dt}\ dt = \int_a^b f(x)\ dx.$$

Proof. The proof is clear from the explicit statement

$$\int_c^d f(\phi(t))\frac{d\phi}{dt}\ dt = \int_a^b f(x)\ dx,$$

[1] A C^1-function is a differentiable function whose derivative is continuous.

since this is nothing else but the substitution formula for definite integrals that you proved in Calculus I! Indeed, since $a = \phi(c)$ and $b = \phi(d)$, we can rewrite this equation as

$$\int_c^d f(\phi(t)) \frac{d\phi}{dt}\ dt = \int_{\phi(c)}^{\phi(d)} f(x)\ dx,$$

which is the substitution formula if we do the change of variables $x = \phi(t)$. ∎

What this means is that we can think of the integral $\int_C \omega$, where we think of $C = [a, b]$ as a "curve" in $\mathbb{R}$, as being defined intrinsically in terms of the geometry of the curve and a choice of orientation. It does not matter how we parametrize the curve, as long as we preserve the orientation: the integral is the same.

Moreover, as stated in the proof of the lemma, reparametrization-invariance is nothing else but the substitution formula for definite integrals. Isn't that cool? What this means is that **the substitution formula for definite integrals is simply the statement that integrals of one-forms over intervals are invariant under orientation-preserving reparametrizations**, or equivalently that they are **invariant under pullback**. Neat! The fact that we need to "transform the differential dx" when we do a substitution is now clear: it comes from the transformation property for one-forms under changes of variables studied in Section 2.3, formulated mathematically in terms of pullback.

3.1.4 Orientation-reversing reparametrizations

In the previous lemma we considered functions ϕ that preserve the orientation. What happens if instead we consider a function such that $\phi(c) = b$ and $\phi(d) = a$, i.e. that maps the left endpoint of the interval $[c, d]$ to the right endpoint of the interval $[a, b]$, and vice-versa? Such a ϕ reverses the orientation from increasing real numbers to decreasing real numbers.

Lemma 3.1.6 Integrals of one-forms over intervals pick a sign under orientation-reversing reparametrizations. *Let ω be a one-form on $U \subseteq \mathbb{R}$ with $[a, b] \subset U$, and $\phi : [c, d] \to [a, b]$ be a function that can be extended to a C^1-function on an open neighborhood of $[c, d]$ and such that $\phi(c) = b$ and $\phi(d) = a$ (it sends the left endpoint of the interval to the right endpoint and vice-versa). Then*

$$\int_{[c,d]} \phi^* \omega = \int_{[a,b]_-} \omega = - \int_{[a,b]} \omega.$$

Proof. From the substitution formula for definite integrals, we know that

$$\int_c^d f(\phi(t)) \frac{d\phi}{dt}\ dt = \int_{\phi(c)}^{\phi(d)} f(x)\ dx = \int_b^a f(x)\ dx = - \int_a^b f(x)\ dx.$$

We recognize the right-hand-side as $- \int_{[a,b]} \omega$, which completes the proof. ∎

It is thus not true that the integral is invariant under all reparametrizations: it is only invariant under reparametrizations that perserve the orientation of the interval. Under reparametrizations that reverse the orientation, the integral changes sign, as expected.

We have thus shown that the integral of a one-form over an interval is both oriented and reparametrization-invariant, our two guiding principles. In the context of definite integrals, orientability reduces to the statement that definite integrals pick a sign when we exchange limits of integration, and reparametrization-invariance to the substitution formula. Neat!

3.1.5 Exercises

1. Consider the one-form $\omega = xe^{x^2}\ dx$. Compute the integral of ω over the interval $[1,2]$, with both choices of orientation.

Solution. We first calculate $\int_{[1,2]}\omega$ with the canonical choice of orientation in the direction of increasing real numbers, i.e. from 1 to 2. We get:

$$\int_{[1,2]_+}\omega = \int_1^2 xe^{x^2}\ dx = \frac{1}{2}\int_1^4 e^u\ du = \frac{1}{2}(e^4 - e),$$

where we used the substitution $u = x^2$. As for the other choice of orientation in the direction of decreasing real numbers, i.e. from 2 to 1, we get:

$$\int_{[1,2]_-}\omega = \int_2^1 xe^{x^2}\ dx = \frac{1}{2}\int_4^1 e^u\ du = -\frac{1}{2}(e^4 - e),$$

which is minus the other integral as expected.

2. Let $\omega = \frac{1}{x^2}\ dx$ be a one-form on $\mathbb{R}_{>0}$, and $\phi : \mathbb{R} \to \mathbb{R}_{>0}$ be given by $\phi(t) = e^t$. Show that we can write

$$\int_{[1,e^3]}\omega = \int_{[0,3]} e^{-t}\ dt.$$

Solution. The map $\phi(t) = e^t$ can be restricted to the interval $[0,3] \subset \mathbb{R}$. We see that its image is the interval $[1,e^3]$, and that the map is injective. Moreover, $\phi(0) = 1$ and $\phi(3) = e^3$, so it preserves orientation. Thus we know that

$$\int_{[0,3]}\phi^*\omega = \int_{[1,e^3]}\omega.$$

We calculate the pullback one-form:

$$\phi^*\omega = \frac{1}{(\phi(t))^2}\phi'(t)\ dt = \frac{1}{e^{2t}}e^t\ dt = e^{-t}\ dt.$$

Therefore, we conclude that

$$\int_{[1,e^3]}\omega = \int_{[0,3]} e^{-t}\ dt.$$

Note that this is just a fancy way of doing a substitution. Indeed, we could write the original integral as follows:

$$\int_{[1,e^3]}\omega = \int_1^{e^3}\frac{1}{x^2}\ dx.$$

We can do the substitution $x = e^t$, and the integral becomes

$$\int_{[1,e^3]}\omega = \int_1^{e^3}\frac{1}{x^2}\ dx = \int_0^3 e^{-t}\ dt = \int_{[0,3]} e^{-t}\ dt,$$

as claimed. Indeed, as we have seen, orienting-preserving reparametrizations of the integral is just the substitution formula for definite integrals.

3. Let $\omega = \sin(x^2)\ dx$ be a one-form on $\mathbb{R}$. TRUE or FALSE:

$$\int_{[1,4]} \omega = 2 \int_{[-2,-1]} t \sin(t^4)\ dt.$$

Solution. To go from the expression in x to the expression in t we need to do a change of variables. More precisely, we consider the smooth function $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $\phi(t) = t^2$. We see that $\phi : [-2,-1] \to [1,4]$, with $\phi(-2) = (-2)^2 = 4$ and $\phi(-1) = (-1)^2 = 1$. This means that it changes the orientation on the interval. So we should get

$$\int_{[-2,-1]} \phi^* \omega = - \int_{[1,4]} \omega.$$

By definition of pullback, we get

$$\phi^* \omega = \sin(t^4) \frac{d\phi}{dt}\ dt = 2t \sin(t^4)\ dt,$$

so we conclude that

$$\int_{[1,4]} \omega = -2 \int_{[-2,-1]} t \sin(t^4)\ dt.$$

Therefore the statement is FALSE.

For fun, let us check that this is consistent with what we expect from the substitution formula. Recall that the substitution formula tells us that

$$\int_c^d f(\phi(t)) \frac{d\phi}{dt}\ dt = \int_{\phi(c)}^{\phi(d)} f(x)\ dx.$$

In our case, this means that

$$2 \int_{-2}^{-1} t \sin(t^4)\ dt = \int_4^1 \sin(x^2)\ dx.$$

We recognize the left-hand-side as

$$\int_{[-2,-1]} \phi^* \omega,$$

and the right-hand-side as

$$- \int_{[1,4]} \omega,$$

which is consistent with what we wrote above.

3.2 Parametric curves in $\mathbb{R}^n$

In the previous section we showed how one-forms can be integrated over intervals in $\mathbb{R}$. Our goal is to generalize this construction to curves in $\mathbb{R}^n$ (we will focus on $\mathbb{R}^2$ and $\mathbb{R}^3$ in this course). To this end, we must first study in more details parametric curves in $\mathbb{R}^n$, and the concept of orientation of a parametric curve.

Objectives

You should be able to:

- Define parametric curves in $\mathbb{R}^n$.
- Determine the image of a parametric curve, or find a parametrization for a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Determine the tangent vector of a parametric curve.
- Define the orientation of a parametric curve.
- Use different parametrizations for the same curve.
- Determine whether a reparametrization preserves or reverses orientation.

3.2.1 Parametric curves

We start by recalling the definition of parametric curves.

Definition 3.2.1 Parametric curves. A **parametric curve** in $\mathbb{R}^n$ is a vector-valued function

$$\begin{aligned}\alpha : [a,b] &\to \mathbb{R}^n\\ t &\mapsto \alpha(t) = (x_1(t), \ldots, x_n(t)),\end{aligned}$$

such that:

1. α can be extended to a C^1-function[1] on an open neighborhood of $[a,b]$;
2. $\alpha'(t) \neq \mathbf{0}$ for all $t \in [a,b]$;
3. if $\alpha(s) = \alpha(t)$ for any two distinct $s,t \in [a,b]$, then $s,t \in \{a,b\}$. [2] In other words, the map α is injective everywhere except possibly at the end points a and b.

The image of α, which we will denote by $C = \alpha([a,b])$, is a one-dimensional subspace of $\mathbb{R}^n$, which is the curve itself. We say that the parametric curve is **smooth** if α can be extended to a smooth (C^∞) function on an open neighborhood of $[a,b]$. ♢

In words, what we are doing is mapping the interval $[a,b]$ to a one-dimensional subspace $C \subset \mathbb{R}^n$, which is the curve we are interested in. The map gives us a way of doing calculus on the curve, by mapping points on the interval, where we can do calculus easily, to points on the curve, which is a more difficult object to work with. This will prove important to define integration along curves: we will pullback using the parametrization to rephrase integration along a curve into integration along the interval $[a,b]$, which we know how to do.

Properties 1 and 2 in Definition 3.2.1 are important to ensure that the image curve C does not have kinks or corners (see for instance Exercise 3.2.6.6). Property 2 also ensures

[1] A C^1-function is a differentiable function whose derivative is continuous.

[2] Here, the standard notation $\{a,b\}$ means the set with two elements a and b. This should not be confused with $[a,b]$, which means the closed interval from a to b, and (a,b), which means the open interval from a to b.

that a parametrization induces a well defined orientation on the curve, as we will see.

Property 3 in Definition 3.2.1 imposes that the map α is injective everywhere except possibly at the endpoints a and b of the interval, which could be mapped to the same point. This ensures that the image curve does not cross itself, and that the parametrization does not go through the same path multiple times (for instance, going around a circle two times).

Because of Property 3, we can distinguish between two types of parametric curves, depending on whether the image curve has endpoints or not.

Definition 3.2.2 Closed parametric curves. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with image $C = \alpha([a, b]) \subset \mathbb{R}^n$.

1. If $\alpha(a) = \alpha(b)$, then we say that the parametric curve is **closed**, as the image curve has no endpoints (it is a loop).

2. If $\alpha : [a, b] \to \mathbb{R}^n$ is injective, we call the set $\partial C = \{\alpha(a), \alpha(b)\}$ consisting only in the endpoints of C the **boundary** of the curve, which we denote by ∂C.

$\Diamond$

Example 3.2.3 Parametrizing the unit circle. Consider the function $\alpha : [0, 2\pi] \to \mathbb{R}^{\nvDash}$ given by

$$\alpha(\theta) = (\cos\theta, \sin\theta).$$

It is easy to see that the image $\alpha([a, b]) \subset \mathbb{R}^2$ is the unit circle $x^2 + y^2 = 1$. Let us check that the conditions in Definition 3.2.1 are satisfied:

1. α is certainly a C^1-function on $\mathbb{R}$; in fact, it is a smooth function, so this is a smooth parametric curve.

2. $\alpha'(t) = (-\sin\theta, \cos\theta)$ is never zero over the interval $[0, 2\pi]$.

3. The only two values of $\theta \in [0, 2\pi]$ that have the same image are $\theta = 0$ and $\theta = 2\pi$, the endpoints of the interval.

Because of the last statement above, this is an example of a closed parametric curve. $\square$

3.2.2 The tangent or velocity vector

The curve C itself, as a subset of $\mathbb{R}^n$, is just a bunch of points. However, the parametrization $\alpha : [a, b] \to \mathbb{R}^n$ gives us more information. One can think of it as parametrizing the trajectory of a particle moving along the curve: the particle starts at the point $\alpha(a) \in \mathbb{R}^n$, and moves along the curve until it reaches the endpoint $\alpha(b) \in \mathbb{R}^n$. As such, we can define a velocity vector, giving the velocity of the particle as it moves along the curve following the given parametrization. Geometrically, the velocity vector is nothing but the tangent vector to the curve at a given point.

Definition 3.2.4 The tangent vector to a parametric curve. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with the map $\alpha(t) = (x_1(t), \ldots, x_n(t))$. The **tangent vector**, or **velocity vector** T is a vector-valued function

$$\mathbf{T} : [a, b] \to \mathbb{R}^n$$

$$t \mapsto \mathbf{T}(t) = \alpha'(t) = (x_1'(t), \ldots, x_n'(t)).$$

In other words, we simply differentiate the component functions of the vector-valued function α. As $\mathbf{T}$ is a vector-valued function, it assigns a vector to every point on the curve C, namely the tangent vector to the curve at that point. ◊

3.2.3 Orientation of a parametric curve

If we think of the parametrization as giving the trajectory of a particle moving along the curve, the tangent vector gives the velocity of the particle at every point on the curve. As we assume that $\alpha'(t)$ is never zero in the definition of parametric curves Definition 3.2.1, we see that the particle travels along the curve without stopping. Moreover, the velocity vector defines a "direction of travel" along the curve C. Thus the parametrization of a curve naturally gives C an orientation, corresponding to the direction of travel. More precisely, the orientation is specified by the tangent vector.

Definition 3.2.5 Orientation of a curve. The **orientation** of a curve $C \subset \mathbb{R}^n$ is given by a choice of direction on the curve, which can be represented geometrically by an arrow on the curve. There are two distinct choices of orientation on a curve: either the arrow is in one direction or it is in the other. ◊

Note that this naturally generalizes the orientation of an interval that we defined in Definition 3.1.3.

Now we see that parametrizing a curve naturally induces a choice of orientation.

Lemma 3.2.6 Parametric curves are oriented. *Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with $C = \alpha([a, b]) \subset \mathbb{R}^n$. Then the tangent vector naturally induces an orientation on C, with the direction given by the direction of the tangent vector at each point on the curve.*

Proof. The key here is that, according to the definition of parametric curves Definition 3.2.1, the tangent vector never vanishes, and it varies continuously. In other words, the velocity vector is never zero. Thus a particle traveling along the curve cannot turn around, as it would first have to stop as the velocity vector varies continuously, but it cannot stop. So the particle always travel in the same direction along the curve, which defines an orientation on the curve. ■

Remark 3.2.7 Another way of thinking about the fact that a parametrization induces an orientation on the curve is that we can think of the parametrization as not only mapping an interval $[a, b]$ to a curve $C \subset \mathbb{R}^n$, but also as mapping the orientation. When we defined parametric curves in Definition 3.2.1, we could think of the domain of α as being not only an interval but an interval with a choice of orientation; we then always assume that the domain of α is given the canonical orientation (in the direction of increasing real numbers). Because of property 2 in Definition 3.2.1, the canonical orientation on the interval is then unambiguously mapped to a choice of direction on the image curve $\alpha([a, b]) = C$, which is the induced orientation on the curve.

Example 3.2.8 Parametrizing the unit circle counterclockwise. Let us go back to our parametrization of the unit circle in Example 3.2.3. We have the function $\alpha : [0, 2\pi] \to \mathbb{R}^{\nvDash}$ given by

$$\alpha(\theta) = (\cos\theta, \sin\theta).$$

We know that its image is the unit circle $x^2 + y^2 = 1$. What is the induced orientation on the circle? We calculate the tangent vector:

$$\mathbf{T}(\theta) = (x'(\theta), y'(\theta)) = (-\sin\theta, \cos\theta).$$

Now consider $\theta = 0$. The parametrization maps this point to $\alpha(0) = (1, 0)$, so this is the point with coordinates $(1, 0)$ on the unit circle. As for the tangent vector, we see that $\mathbf{T}(0) = (0, 1)$, and hence the tangent vector at the point $(1, 0)$ on the unit circle is pointing upwards. This tells us that we are moving along the curve in a counterclockwise direction, which is the induced orientation on the unit circle. □

3.2.4 Orientation-preserving reparametrizations

Given a curve $C \subset \mathbb{R}^n$, there isn't a unique choice of parametrization; the curve can be parametrized in many different ways. For instance, let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with $\alpha(t) = (x_1(t), \ldots, x_n(t))$. Now suppose that we think of the parameter t as a function of a new parameter u, that is, $t = t(u)$. Our parametrization then becomes $\alpha(t(u)) = (x_1(t(u)), \ldots, x_n(t(u)))$. Assuming that $t(u)$ is chosen appropriately, this may define a new parametrization $\beta(u) = (X_1(u), \ldots, X_n(u)) = (x_1(t(u)), \ldots, x_n(t(u)))$. Going from t to u is what is called "reparametrizing the curve".

We can be a little more precise, using the concept of pullbacks introduced in Definition 2.3.3.

Lemma 3.2.9 Reparametrizations of a curve. *Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with $\alpha(t) = (x_1(t), \ldots, x_n(t))$ and $\alpha([a, b]) = C$. Let $\phi : [c, d] \to [a, b]$ be a function that can be extended to a C^1-function on an open neighbhorood of $[c, d]$. Assume that ϕ is injective, and that $\phi'(u) \neq 0$ for all $u \in [c, d]$. The pullback*

$$\begin{aligned} \phi^*\alpha : [c, d] &\to \mathbb{R}^n \\ u \mapsto &(\phi^* x_1(u), \ldots, \phi^* x_n(u)) = (x_1(\phi(u)), \ldots, x_n(\phi(u))) \end{aligned}$$

is another parametrization of the same curve C.

Proof. First, it is clear that $\alpha([a, b]) = \phi^*\alpha([c, d])$, as we are just composing the functions x and y with ϕ, and thus both α and $\phi^*\alpha$ have the same image curve. But to check that $\phi^*\alpha$ is a parametrization, we need to make sure that the three conditions in Definition 3.2.1 are satisfied.

Property one is certainly satisfied, since it is assumed that ϕ can be extended to a C^1-function on an open neighbhorood of $[c, d]$. Property two is also satisfied, since

$$(\phi^*\alpha)'(u) = \left(\frac{dx_1}{d\phi}\frac{d\phi}{du}, \ldots, \frac{dx_n}{d\phi}\frac{d\phi}{du} \right),$$

and by assumption $\frac{d\phi}{du}$ never vanishes on $[c, d]$. As for property three, it follows since ϕ is assumed to be injective. ■

Remark 3.2.10 We note that since $\frac{d\phi}{du}$ is continuous, and is never zero on $[c, d]$, then it is either everywhere positive on $[c, d]$, or everywhere negative.

Since a parametric curve induces a choice of orientation on the curve, and there are only two possible choices of orientation, it will be important to distinguish between reparametrizations that preserve the induced orientation, and those that reverse it.

Lemma 3.2.11 Orientation-preserving reparametrizations. *Consider a reparametrization as in Lemma 3.2.9, and see Remark 3.2.10. If $\frac{d\phi}{du} > 0$ for all $u \in [c,d]$, then the two parametrizations α and $\phi^*\alpha$ induce the same orientation, and we call the reparametrization* ***orientation-preserving****. If $\frac{d\phi}{du} < 0$ for all $u \in [c,d]$, then the two parametrizations α and $\phi^*\alpha$ induce opposite orientations, and we call the reparametrization* ***orientation-reversing****.*

Proof. We simply need to compare the tangent vectors. Let T_α be the tangent vector associated to the parametrization α, and $T_{\phi^*\alpha}$ be the tangent vector associated to $\phi^*\alpha$. We have:

$$T_{\phi^*\alpha}(u) = \frac{d\phi}{du}\left(x_1'(\phi(u)), \ldots, x_n'(\phi(u))\right) = \frac{d\phi}{du} T_\alpha(\phi(u)).$$

It is then clear that if $\frac{d\phi}{du} > 0$, the orientation is preserved, while if $\frac{d\phi}{du} < 0$ is is reversed. ■

Example 3.2.12 Two parametrizations of the unit circle. We already saw in Example 3.2.8 a parametrization of the unit circle that induces a counterclockwise orientation, namely $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with $\alpha(\theta) = (\cos\theta, \sin\theta)$. Now consider a second parametrization of the unit circle $\beta : [-\frac{3\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2$ with $\beta(t) = (\sin t, \cos t)$. What orientation is β inducing? The tangent vector reads:

$$\mathbf{T}_\beta(t) = (\cos t, -\sin t).$$

Consider $t = \pi/2$. The parametrization maps this point to the point $\beta(\pi/2) = (1, 0)$ on the unit circle. The tangent vector is $\mathbf{T}_\beta(\pi/2) = (0, -1)$, and hence it points downwards. We conclude that β is inducing a clockwise orientation on the circle, which is the opposite of our original parametrization α.

Let us now formulate this in the language of reparametrizations as above. Consider the function $\phi : [-\frac{3\pi}{2}, \frac{\pi}{2}] \to [0, 2\pi]$ given by $\phi(t) = \frac{\pi}{2} - t$. This function is injective, and $\phi'(t) = -1$ which is of course never zero. The pullback $\phi^*\alpha : [-\frac{3\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2$ is given by

$$\phi^*\alpha(t) = (\cos(\frac{\pi}{2} - t), \sin(\frac{\pi}{2} - t)) = (\sin t, \cos t),$$

which is our second parametrization β. Since $\phi'(t) < 0$, we expect α and $\beta = \phi^*\alpha$ to induce opposite orientation, which is exactly what we observed. □

3.2.5 Piecewise parametric curves

To end this section, we note that it will sometimes be useful to consider unions of parametric curves as defined in Definition 3.2.1. This is because our definition is fairly restrictive. It would not allow for curves with kinks or corners, for instance, since we impose that $\alpha'(t)$ is never zero. Also, since we require that α is injective except possibly at the endpoints, it would not allow for curves with self-intersection. To deal with these cases, all we need to do is consider the union $C_1 \cup \ldots \cup C_n$ of a finite number of curves with parametrizations, and such that two distinct components can only have one or two points in common (their endpoints). For instance, in the case of a curve with corners, we will treat it as a union of parametric curves, where each curve starts where the previous one ends.

If a piecewise parametric curve is the union of a number of parametric curves, and each parametric curve is smooth, we call the piecewise parametric curve **piecewise smooth**.

Remark 3.2.13 We add one more piece of notation. It will be useful to distinguish between curves that have self-intersection and those that do not. We say that a curve that doesn't intersect itself (except possibly at the endpoints) is **simple**. With our definition of parametric curves Definition 3.2.1, the image of a parametric curve will be always be simple, as it cannot self-intersect.

Non-simple curves can be studied using piecewise parametric curves, as any non-simple curve can be broken into a number of simple curves.

Example 3.2.14 Parametrizing a triangle. Consider the triangle with vertices $A = (0, 0)$, $B = (0, 1)$ and $C = (1, 0)$. We cannot parametrize it as a single parametric curve according to Definition 3.2.1, since we cannot find a parametrization α that has a non-vanishing tangent vector at the vertices of the triangle. Instead, we split it into the union of three parametric curves, corresponding to the three edges of the triangle:

$$\begin{aligned} \alpha_1 :&[0,1] \to \mathbb{R}^2, \qquad & \alpha_1(t) &= (0,t),\\ \alpha_2 :&[0,1] \to \mathbb{R}^2, \qquad & \alpha_2(t) &= (t,1-t),\\ \alpha_3 :&[0,1] \to \mathbb{R}^2, \qquad & \alpha_3(t) &= (1-t,0). \end{aligned}$$

α_1 parametrizes the edge AB, α_2 the edge BC, and α_3 the edge CA. As each parametric curve is smooth, the union defines a piecewise smooth parametric curve. □

3.2.6 Exercises

1. Find a parametrization for the straight line between the points $(0, 1, 1)$ and $(2, 3, 3)$ in $\mathbb{R}^3$.

Solution. We are given two points on the line. The vector $\mathbf{d}$ whose direction is parallel to the line is given by $\mathbf{d} = (2, 3, 3) - (0, 1, 1) = (2, 2, 2)$. So we can write an equation for the line as

$$\mathbf{r}(t) = (0,1,1) + t(2,2,2), \qquad 0 \leq t \leq 1.$$

In the language of this section, this gives us a parametrization of the line, in the form of a map:

$$\begin{aligned} \alpha :&[0,1] \to \mathbb{R}^3\\ &t \mapsto \alpha(t) = (2t, 1+2t, 1+2t). \end{aligned}$$

2. Express the upper half of a circle of radius 3 and centered at the point $(1, 0)$ as a parametric curve. Determine the orientation induced by your parametrization.

Solution. The circle of radius 3 and centered at the point $(1, 0)$ has equation

$$(x-1)^2 + y^2 = 9.$$

It is easy to find a parametrization for the circle. We can take, for instance,

$$x - 1 = 3\cos t, \qquad y = 3\sin t, \qquad 0 \leq t \leq 2\pi.$$

We want only the upper half of the circle though, so we need to restrict to $y \geq 0$. This amounts to restricting the domain of our parametrization to $0 \leq t \leq \pi$. The resulting parametrization can be written as the map $\alpha : [0, \pi] \to \mathbb{R}^2$ with $\alpha(t) = (3\cos t + 1, 3\sin t)$.

What is the induced orientation? The tangent vector to our parametric curve is $\mathbf{T}(t) = (-3\sin t, 3\cos t)$. Pick a point on the circle, say $(4, 0)$. This corresponds to $t = 0$. At this point, the tangent vector is $\mathbf{T}(0) = (0, 3)$, which is pointing upwards. This means that our parametrization induces a counterclockwise orientation around the circle.

3. Consider the parametric curve $\alpha : [0, 4\pi] \to \mathbb{R}^3$, $\alpha(t) = (\cos t, \sin t, 4t)$. What is the shape of the image curve $C = \alpha([0, 4\pi]) \subset \mathbb{R}^3$? What is the induced orientation?

Solution. Let us write $\alpha(t) = (x(t), y(t), z(t))$. We see that $x^2(t) + y^2(t) = \cos^2 t + \sin^2 t = 1$, so all points on the curve C lie on the cylinder $x^2 + y^2 = 1$ with radius 1. Moreover, as t increases, $z(t) = 4t$ increases linearly. So the curve is an helix on the cylinder with radius 1 centred around the z-axis.

The curve starts at the point $\alpha(0) = (1, 0, 0)$ on the cylinder, and ends at the point $\alpha(4\pi) = (1, 0, 16\pi)$. As t runs from 0 to 4π, we see that the curve goes twice around the cylinder. Moreover, the tangent vector is $\mathbf{T}(t) = (-\sin t, \cos t, 4)$; in particular, its z-coordinate is always positive, which means that the tangent vector is always pointing upwards. We conclude that the induced orientation is going upwards along the helix.

4. Suppose that a particle is moving along the parametric curve $\alpha : [0, \pi] \to \mathbb{R}^3$ with $\alpha(t) = (\sin(t^2), \cos(t), t)$. Find its velocity at $t = \pi$.

Solution. The velocity vector is $\mathbf{v}(t) = (2t\cos(t^2), -\sin(t), 1)$. At $t = \pi$, we get $\mathbf{v}(\pi) = (2\pi\cos(\pi^2), 0, 1)$. This is the velocity of the particle at $t = \pi$, which is of course a vector as the particle is moving in three-dimensional space.

5. Consider the curve that is the intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = x^2 - y^2$ in $\mathbb{R}^3$. Find a parametrization for the curve. Is it a closed curve?

Solution. A point on the cylinder $x^2 + y^2 = 1$ can be parametrized by $(x, y, z) = (\cos t, \sin t, z)$, with $0 \leq t < 2\pi$ and $z \in \mathbb{R}$. But we want the curve to lie on the surface $z = x^2 - y^2$, so z is fixed as $z = \cos^2 t - \sin^2 t$. We thus get a parametric expression for a point on the curve as $(x(t), y(t), z(t)) = (\cos t, \sin t, \cos^2 t - \sin^2 t)$, with $0 \leq t < 2\pi$. To rewrite this as a parametric curve in the language of this section, we include the endpoint $t = 2\pi$. We get the parametric curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ with $\alpha(t) = (\cos t, \sin t, \cos^2 t - \sin^2 t)$. This is a closed curve, since $\alpha(0) = (1, 0, 1)$ and $\alpha(2\pi) = (1, 0, 1)$, i.e. the starting and ending points coincide.

6. Consider the map $\alpha : [-1, 1] \to \mathbb{R}^2$ with $\alpha(t) = (x(t), y(t))$, where

$$x(t) = \begin{cases} t^2 & \text{for } 0 \leq t \leq 1 \\ -t^2 & \text{for } -1 \leq t < 0 \end{cases}, \qquad y(t) = t^2.$$

Show that it is not a parametric curve, according to Definition 3.2.1. What does the image $C = \alpha([-1, 1]) \subset \mathbb{R}^2$ look like?

Solution. At first one may think that this is valid parametric curve. The map α is injective, so Property 3 is satisfied. Since its derivative is $\alpha'(t) = (x'(t), y'(t))$ with

$$x'(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq 1 \\ -2t & \text{for } -1 \leq t < 0 \end{cases}, \qquad y'(t) = 2t,$$

$\alpha'(t)$ exists and is continuous for all $t \in \mathbb{R}$, and thus Property 1 is also satisfied (note that α cannot be extended to a smooth function however, since $x'(t)$ is not differentiable at $t = 0$, but that's ok, it doesn't have to). However, the problem is with Property 2: we see that $\alpha'(0) = (0, 0)$, with $0 \in [-1, 1]$; thus Property 2 is not satisfied. We conclude that this is not a parametric curve.

The image curve $C = \alpha([-1, 1])$ is the set of points in $\mathbb{R}^2$ satisfying the equation $y = |x|$ between $(-1, 1)$ and $(1, 1)$, which has a corner at the origin. This example highlights one of the reasons why Property 2 is there in Definition 3.2.1. If it wasn't there, this means that we could find a parametrization for the curve $y = |x|$; but we do not want the image curve to have kinks or corners. Property 2 ensures that we cannot find a parametrization for the whole curve $y = |x|$ between $(-1, 1)$ and $(1, 1)$ at once.

This is not to say however that we cannot deal with this curve. Just like for the triangle, the idea is to consider it as a piecewise parametric curve. I.e., we realize the line segment from $(-1, 1)$ to $(0, 0)$ as a (smooth) parametric curve, the line segment from $(0, 0)$ to $(1, 1)$ as another (smooth) parametric curve, and we take the union of the two parametric curves.

3.3 Line integrals

We are finally ready to define the integral of a one-form over a curve in $\mathbb{R}^n$. Our strategy is to start with a parametrization for the curve, and then use the parametrization to pullback the integrand to the interval $[a, b]$, over which we know how to integrate. In other words, we reduce a complicated problem to something that we already know how to solve!

However, ultimately we would like our integral to be independent of our choice of parametrization. We show that the integral is indeed invariant under orientation-preserving reparametrizations, and thus can be understood as an object defined solely in terms of the geometry of a curve and a choice of orientation. We also show that the integral changes sign under orientation-reversing reparametrizations, as expected.

Objectives

You should be able to:

- Determine the pullback of a one-form along a parametric curve in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Reformulate the pullback of a one-form along a parametric curve in terms of the associated vector fields.
- Define the line integral of a one-form along a parametric curve in $\mathbb{R}^2$ and $\mathbb{R}^3$, and evaluate it.
- Rewrite the definition of line integrals in terms of the associated vector fields.

- Show that line integrals are invariant under orientation-preserving reparametrizations of the curve.
- Show that line integrals change sign under reparameterizations of the curve that reverse its orientation.

3.3.1 The pullback of a one-form along a parametric curve

Consider a one-form ω on an open subset U of $\mathbb{R}^n$. Suppose that C is a curve which is contained in U. Our goal is to define an integral of the form "$\int_C \omega$" for the integral of the one-form ω along the curve C (with a choice of orientation on C). However, this is not so obvious, as it is not clear what it means to "integrate along a curve". To make sense of this, we use a parametrization for C, which is a map $\alpha : [a, b] \to \mathbb{R}^n$. The idea is to use the powerful concept of pullback, which we studied in Section 2.4, to pull back the one-form from U to the interval $[a, b]$, and then we can integrate it, as we know how to integrate a one-form over an interval: Definition 3.1.1. Neat!

We can apply the general expression for the pullback of a one-form obtained in Lemma 2.4.5 to the case where ϕ is replaced by a parametric curve $\alpha : [a, b] \to \mathbb{R}^n$. Let us write explicit expressions for the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let $\alpha : [a, b] \to \mathbb{R}^2$ be a parametric curve as in Definition 3.2.1, with $\alpha(t) = (x(t), y(t))$, and let $\omega = f(x, y)\ dx + g(x, y)\ dy$ be a one-form on an open subset $U \subseteq \mathbb{R}^2$ containing $C = \alpha([a, b])$. Then $\alpha^*\omega$ is a one-form on an open subset $V \subseteq \mathbb{R}$ containing $[a, b]$ defined by

$$\alpha^*\omega = \left(f(\alpha(t))\frac{dx}{dt} + g(\alpha(t))\frac{dy}{dt} \right)\ dt.$$

Similarly, given a parametric curve $\alpha : [a, b] \to \mathbb{R}^3$, with $\alpha(t) = (x(t), y(t), z(t))$, and a one-form $\omega = f(x, y, z)\ dx + g(x, y, z)\ dy + h(x, y, z)\ dz$ on $U \subseteq \mathbb{R}^3$, the pullback $\alpha^*\omega$ is given by:

$$\alpha^*\omega = \left(f(\alpha(t))\frac{dx}{dt} + g(\alpha(t))\frac{dy}{dt} + h(\alpha(t))\frac{dz}{dt} \right)\ dt.$$

Example 3.3.1 Pulling back along a circle. Consider the counterclockwise parametrization of the unit circle introduced in Example 3.2.3, given by the function $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with $\alpha(\theta) = (\cos\theta, \sin\theta)$. Let $\omega = x^2 y\ dx + e^y\ dy$ be a one-form on $\mathbb{R}^2$. We can pull it back along the parametrized unit circle to get a new one-form on $\mathbb{R}$:

$$\begin{aligned} \alpha^*\omega &= \left((\cos\theta)^2 \sin\theta \frac{d}{d\theta}\cos\theta + e^{\sin\theta}\frac{d}{d\theta}\sin\theta \right)\ d\theta \\ &= \left(-\cos^2\theta\sin^2\theta + \cos\theta e^{\sin\theta} \right)\ d\theta. \end{aligned}$$

□

As is becoming customary, we can translate between the language of differential forms and the language of vector fields. If $\mathbf{F}$ is the vector field associated to ω, and α is a parametric curve in $\mathbb{R}^n$, then the pullback one-form $\alpha^*\omega$ can be written as

$$\alpha^*\omega = (\mathbf{F}(\alpha(t)) \cdot \mathbf{T}(t))\ dt,$$

where $\mathbf{T}(t)$ is the tangent vector to the parametric curve, and $\cdot$ denotes the dot product of vectors.

3.3.2 The definition of line integrals

We are now ready to define the integral of a one-form along a parametric curve: we pull back the one-form to the interval $[a,b]$ and integrate.

Definition 3.3.2 (Oriented) line integrals. Let ω be a one-form on an open subset U of $\mathbb{R}^n$, and let $\alpha : [a,b] \to \mathbb{R}^n$ be a parametric curve whose image $C \subset U$. We define the **line integral of ω along α** as follows:

$$\int_\alpha \omega = \int_{[a,b]} \alpha^* \omega,$$

where the integral on the right-hand-side is defined in Definition 3.1.1.[1]

Explicitly, focusing on $\mathbb{R}^2$ for simplicity, if $\omega = f(x,y)\ dx + g(x,y)\ dy$, and $\alpha(t) = (x(t), y(t))$, the integral reads:

$$\int_\alpha \omega = \int_a^b \left(f(\alpha(t)) \frac{dx}{dt} + g(\alpha(t)) \frac{dy}{dt} \right)\ dt.$$

A similar expression of course holds in $\mathbb{R}^3$ as well.

Such integrals are also called **work integrals** because of physical applications, as we will see in Section 3.5. ♢

What is neat is that on the right-hand-side, we end up with a one-variable definite integral, which we certainly know how to integrate from Calculus I and II. So we have reduced the problem of computing integrals of one-forms along curves to standard definite integrals!

Example 3.3.3 An example of a line integral. Consider the one-form $\omega = y\ dx + \cos(x)\ dy$, and the parametric curve $\alpha : [0,1] \to \mathbb{R}^2$ with $\alpha(t) = (t, t^2)$.

First, for completeness we check that the parametric curve is well defined, according to Definition 3.2.1. α is a smooth function on $\mathbb{R}$; $\alpha'(t) = (1, 2t)$ which is never zero on $\mathbb{R}$, so in particular never zero on $[0,1]$; α is injective over $[0,1]$. Thus α defines a smooth parametric curve, and its image $C = \alpha([0,1])$ has two boundary points at $\alpha(0) = (0,0) \in \mathbb{R}^2$ and $\alpha(1) = (1,1) \in \mathbb{R}^2$. In fact, it is not difficult to see that α is a parametrization of the parabola $y - x^2 = 0$ between $(0,0)$ and $(1,1)$.

We can compute the integral of ω over the parametric curve α. We get:

$$\begin{aligned}
\int_\alpha \omega &= \int_{[0,1]} \alpha^* \omega \\
&= \int_0^1 \left(f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right)\ dt \\
&= \int_0^1 \Big((t^2)(1) + \cos(t)(2t) \Big)\ dt \\
&= \left(\frac{t^3}{3} + 2(t\sin(t) + \cos(t)) \right) \Big|_0^1 \\
&= \frac{1}{3} + 2\sin(1) + 2\cos(1) - 2
\end{aligned}$$

[1] In particular, the integral on the right-hand-side uses the canonical orientation on the interval $[a,b]$, which is consistent with the orientation on the image curve C induced by the parametrization.

$$= -\frac{5}{3} + 2\sin(1) + 2\cos(1).$$

□

Remark 3.3.4 Line integrals over piecewise parametric curves. We note that we can easily generalize the definition of line integrals to piecewise parametric curves, as in Subsection 3.2.5. If the parametric curve is defined as a union of parametric curves, then to integrate along the curve we simply add up the integrals over the pieces.

3.3.3 Reparametrization-invariance and orientability of line integrals

We defined line integrals in terms of a parametric curve $\alpha : [a,b] \to \mathbb{R}^n$, but in the end we would like the integral to be defined intrinsically in terms of the geometry of the image curve and a choice of orientation. To show that this is the case, we now show that our definition is invariant under orientation-preserving reparametrizations. We also show that it changes sign under orientation-reversing reparametrizations, which shows that line integrals are in fact oriented, as we want.

Lemma 3.3.5 Line integrals are invariant under orientation-preserving reparametrizations. *Let ω be a one-form on an open subset U of $\mathbb{R}^n$, and let $\alpha : [a,b] \to \mathbb{R}^n$ be a parametric curve whose image $C \subset U$. Let $\phi : [c,d] \to [a,b]$ be as in Lemma 3.2.9, so that $\phi^*\alpha = \alpha \circ \phi : [c,d] \to \mathbb{R}^n$ is a reparametrization of the curve.*

1. *If ϕ preserves orientation, as defined in Lemma 3.2.11, then*

$$\int_\alpha \omega = \int_{\phi^*\alpha} \omega.$$

2. *If ϕ reverses orientation, then*

$$\int_\alpha \omega = -\int_{\phi^*\alpha} \omega.$$

In other words, the integral is invariant under orientation-preserving reparametrizations, and changes sign under orientation-reversing reparametrizations. So we can really think of the line integral as being defined intrinsically in terms of the image curve C itself, with a choice of orientation.

Proof. To prove this statement, let us first rewrite the integrals as integrals over intervals, using Definition 3.3.2. The integral on the left-hand-side is:

$$\int_\alpha \omega = \int_{[a,b]} \alpha^*\omega.$$

As for the second integral, we are integrating over the parametric curve $\alpha \circ \phi : [c,d] \to [a,b] \to \mathbb{R}^n$. So we can write

$$\int_{\phi^*\alpha} \omega = \int_{[c,d]} (\alpha \circ \phi)^*\omega.$$

However, from Exercise 2.4.3.6 we know that pulling back through the chain of maps $[c,d] \xrightarrow{\phi} [a,b] \xrightarrow{\alpha} \mathbb{R}^n$ is the same thing as doing it in two steps: first pulling back via α, and then via ϕ.

In other words, $(\alpha \circ \phi)^* \omega = \phi^*(\alpha^* \omega)$, and we can write

$$\int_{\phi^* \alpha} \omega = \int_{[c,d]} \phi^*(\alpha^* \omega).$$

The statement then is about the relation between $\int_{[a,b]} \alpha^* \omega$ and $\int_{[c,d]} \phi^*(\alpha^* \omega)$. But we already studied such questions before in Lemma 3.1.5 and Lemma 3.1.6! Indeed, now that we wrote everything in terms of integrals of one-forms over intervals, we are back in the realm of Section 3.1. From Lemma 3.1.5 we know that the two integrals will be equal if $\phi(c) = a$ and $\phi(d) = b$, i.e. ϕ preserves the order of the endpoints of the interval (recall that invariance in this case is simply the statement of the substitution formula for definite integrals). If instead ϕ exchanges the order of the endpoints (i.e. $\phi(c) = b$ and $\phi(d) = a$), then by Lemma 3.1.6 the integrals differ by a sign.

But if $\phi^* \alpha$ is orientation-preserving, then $\phi'(u) > 0$ by Lemma 3.2.11, and so ϕ is a strictly increasing function, which means that it must map $c \to a$ and $d \to b$, as $c \leq d$ and $a \leq b$. Therefore the integrals are equal. While if $\phi^* \alpha$ is orientation-reversing, then $\phi'(u) < 0$, and so ϕ is a strictly decreasing function, which means that it must map $c \to b$ and $a \to d$. Therefore the integrals differ by a sign. This completes the proof of the lemma. ■

Great, our two principles of orientability and reparametrization-invariance are fulfilled for line integrals!

Example 3.3.6 How line integrals change under reparametrizations. Let us consider the integral from Example 3.3.3 again. We consider the one-form $\omega = y\, dx + \cos(x)\, dy$, and the parametric curve $\alpha : [0, 1] \to \mathbb{R}^2$ with $\alpha(t) = (t, t^2)$.

Let us define two new parametrizations for the same curve. First, we define $\beta : [0, \ln(2)] \to \mathbb{R}^2$ with $\beta(t) = (e^t - 1, (e^t - 1)^2)$, and $\gamma : [-1, 0] \to \mathbb{R}^2$ with $\gamma(t) = (-t, t^2)$. It is easy that α, β and γ are all parametrizations of the parabola $y - x^2 = 0$ between $(0, 0)$ and $(1, 1)$.

Looking at the tangent vectors, we get $\alpha'(t) = (1, 2t)$, $\beta'(t) = (e^t, 2(e^t - 1)e^t)$, and $\gamma'(t) = (-1, 2t)$. Are those all inducing the same orientation on the curve? Let us look at the direction of the tangent vector at $(0, 0)$, which corresponds to $t = 0$ for all three parametrizations. We have $\alpha'(0) = (1, 0)$, $\beta'(0) = (1, 0)$, and $\gamma'(0) = (-1, 0)$. So α and β induce the same orientation on the curve (from $(0, 0)$ to $(1, 1)$), while γ induces the opposite orientation. Therefore, we expect the integral of ω over α and β to both give the same answer, while the integral over γ should pick a sign.

Let us do the calculation for fun. The integral over α was already performed in Example 3.3.3. For β, we get:

$$\begin{aligned}
\int_\beta \omega &= \int_{[0,\ln(2)]} \beta^* \omega \\
&= \int_0^{\ln(2)} \left(f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right) \, dt \\
&= \int_0^{\ln(2)} \left((e^t - 1)^2 e^t + 2e^t (e^t - 1) \cos(e^t - 1) \right) \, dt \\
&= \int_0^1 \left(u^2 + 2u \cos(u) \right) \, du,
\end{aligned}$$

where we did the substitution $u = e^t - 1$. But this is the same integral as in Example 3.3.3, so we get the same result indeed.

As for γ, we get:

$$\begin{aligned}\int_\gamma \omega &= \int_{[-1,0]} \gamma^* \omega \\ &= \int_{-1}^0 \left(f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right) \, dt \\ &= \int_{-1}^0 \left(-t^2 + 2t \cos(-t) \right) \, dt \\ &= \int_1^0 \left(u^2 + 2u \cos(u) \right) \, du,\end{aligned}$$

where we did the substitution $u = -t$. By exchanging the limits of integration, we see that this is minus the integral in Example 3.3.3, and thus we get a minus sign as expected. □

3.3.4 Line integrals in terms of vector fields

Now that we know how to integrate one-forms along curves, we can translate the definition in terms of the associated vector fields. This is straightforward, since we saw in Subsection 3.3.1 how to rephrase the pullback of a one-form along a curve in terms of the associated vector field.

Lemma 3.3.7 Line integrals in terms of vector fields. *Let ω be a one-form on an open subset U of $\mathbb{R}^n$, and let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve whose image $C \subset U$. Let $\mathbf{F} : U \to \mathbb{R}^n$ be the vector field associated to ω. Then we can write the line integral as follows:*

$$\int_\alpha \omega = \int_a^b \mathbf{F}(\alpha(t)) \cdot \mathbf{T}(t) \, dt.$$

Proof. This is clear, using the translation established in Subsection 3.3.1. ■

This is how line integrals, or "work integrals", are generally defined in standard vector calculus textbooks.[2] The integrand is justified in the context of the calculation of work in physics (as we will see in Section 3.5, such line integrals can be used to calculate work), but it is not clear why taking the dot product between the vector field and the tangent vector to the parametric curve is the right thing to do in general. In our context, the integrand arises naturally by pulling back the one-form in order to be able to integrate over an interval. So it gives a natural geometric interpretation to these work integrals.

Remark 3.3.8 In standard vector calculus textbooks, such as CLP4, the following notation is often used. Instead of writing $\alpha : [a, b] \to \mathbb{R}^n$ for the parametric curve, the symbol $\mathbf{r} = \mathbf{r}(t)$ is often used, with $a \leq t \leq b$, as if it was the position function of an object moving along the image curve C. Then the tangent (or velocity) vector is written as

$$\mathbf{T} = \frac{d\mathbf{r}}{dt},$$

[2] Note that the symbol $\mathbf{T}(t)$ is sometimes used to denote the normalized or unit tangent vector (i.e. our tangent vector divided by its norm $|\mathbf{T}(t)|$), in which case dt should be replaced by $ds = |\mathbf{T}(t)| \, dt$.

standing for the velocity of the object moving along the curve, and the notation

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}\ dt$$

is used. With this notation, one can rewrite the line integral of a vector field along the curve as

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

The notation makes sense, as we know that the integral is invariant under orientation-preserving reparametrizations, so we can rewrite it in terms of the image curve C itself, with the orientation specified by the direction of travel along the curve. With this notation however one needs to keep in mind that to evaluate the integral we need to compose the vector field with the parametrization to rewrite $\mathbf{F}$ as a function of the parameter t before integrating in t over the interval $[a, b]$.

3.3.5 Exercises

1. Consider the one-form $\omega = xy\ dx + z^2\ dy + z\ dz$ on $\mathbb{R}^3$. Find its pullback along the helix centered around the z-axis and with tangent vector $\mathbf{T}(t) = (-3\sin t, 3\cos t, 4)$, $0 \le t \le 2\pi$, and initial position $\mathbf{r}(0) = (3, 0, 1)$.

Solution. First, we find a parametrization for the helix. Integrating the tangent vector, we know that the parametrization must be given by $\alpha : [0, 2\pi] \to \mathbb{R}^3$ with $\alpha(t) = (3\cos t + A, 3\sin t + B, 4t + C)$ for some constants A, B, C. Using the fact that $\alpha(0) = (3, 0, 1)$, we conclude that $A = 0$, $B = 0$, and $C = 1$. Thus the parametrization is $\alpha(t) = (3\cos t, 3\sin t, 4t + 1)$.

We can then calculate the pullback $\alpha^*\omega$. We get:

$$\begin{aligned}\alpha^*\omega &= \Big((3\cos t)(3\sin t)(-3\sin t) + (4t+1)^2(3\cos t) + (4t+1)(4)\Big)\ dt\\ &= \Big(-27\cos t\sin^2 t + 3(4t+1)^2\cos t + 4(4t+1)\Big)\ dt.\end{aligned}$$

2. Consider the one-form $\omega = \frac{x}{x^2+y^2}\ dx + \frac{y}{x^2+y^2}\ dy + z\ dz$. Explain why you cannot pull it back along the parametric curve $\alpha : [0, 2] \to \mathbb{R}^3$ with $\alpha(t) = (t-1, t-1, t^2)$.

Solution. The key here is to be careful about the domain of definition of the one-form ω. The largest open subset $U \subseteq \mathbb{R}^3$ over which ω is defined is all points in $\mathbb{R}^3$ such that $x^2 + y^2 \neq 0$, so that the denominators in the component functions do not vanish. But $x^2 + y^2 = 0$ if and only if $(x, y) = (0, 0)$, so the domain of definition of ω is $U = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}$. In other words, it consists of all points in $\mathbb{R}^3$ except the points on the z-axis.

To be able to pull back our one-form along the parametric curve α, we must make sure that its image $C = \alpha([0, 2])$ lies within U. Unfortunately, we see that at $t = 1$, $\alpha(1) = (0, 0, 1)$, so the parametric curve intersects the z-axis, which is not in U! Thus we can't pull back along α.

Just for fun, let's see what would happen if we had naively tried to pull back along α.

What we would get is:

$$\alpha^*\omega = \left(\frac{t-1}{2(t-1)^2} + \frac{t-1}{2(t-1)^2} + t^2(2t) \right) \, dt = \left(\frac{1}{t-1} + 2t^3 \right) \, dt.$$

The problem is that this is not defined at $t = 1$, which is part of the interval $[0, 2]$ for the parametric curve. In other words, the result of the pullback is not actually a well defined one-form on an open subset containing $[0, 2]$, since it is not defined at $t = 1$.

3. Find the integral of the one-form $\omega = x \, dx + x \, dy + y \, dz$ in $\mathbb{R}^3$ along one turn clockwise around the circle of radius two in the xy-plane and centered at the origin.

Solution. The circle has equation $x^2 + y^2 = 4$. We parametrize one turn clockwise around the circle with $\alpha : [0, 2\pi] \to \mathbb{R}^3$, $\alpha(t) = (2 \sin t, 2 \cos t, 0)$ (recall that we are in $\mathbb{R}^3$). The pullback of the one-form is

$$\alpha^*\omega = ((2 \sin t)(2 \cos t) + (2 \sin t)(-2 \sin t) + (2 \cos t)(0)) \, dt = 2 \left(\sin(2t) + \cos(2t) - 1 \right) \, dt.$$

The line integral becomes

$$\begin{aligned} \int_{[0,2\pi]} \alpha^*\omega =& 2 \int_0^{2\pi} (\sin(2t) + \cos(2t) - 1) \, dt \\ =& (-\cos(4\pi) + \cos(0)) + (\sin(4\pi) - \sin(0)) - 2(2\pi) \\ =& -4\pi. \end{aligned}$$

4. Consider the vector field $\mathbf{F}(x, y) = (e^y, e^x)$ in $\mathbb{R}^2$. Find its line integral along the oriented curve obtained by first moving from the origin to the point $(0, 1)$, and then from $(0, 1)$ to the point $(4, 0)$ along straight lines.

Solution. This is a piecewise parametric curve, so we need to split it into line segments. For the first segment from $(0, 0)$ to $(0, 1)$, we can write a parametrization as $\alpha_1 : [0, 1] \to \mathbb{R}^2$ with $\alpha_1(t) = (0, t)$. For the second line segment from $(0, 1)$ to $(0, 4)$, we write a parametrization as $\alpha_2 : [0, 1] \to \mathbb{R}^2$ with $\alpha_2(t) = (4t, 1 - t)$. The tangent vectors are:

$$\mathbf{T}_1(t) = (0, 1), \qquad \mathbf{T}_2(t) = (4, -1).$$

We thus get:

$$\mathbf{F}(x(t), y(t)) \cdot \mathbf{T}_1(t) = 1, \qquad \mathbf{F}(x(t), y(t)) \cdot \mathbf{T}_2(t) = 4e^{1-t} - e^{4t}.$$

The line integral becomes:

$$\begin{aligned} &\int_{[0,1]} \mathbf{F}(x(t), y(t)) \cdot \mathbf{T}_1(t) \, dt + \int_{[0,1]} \mathbf{F}(x(t), y(t)) \cdot \mathbf{T}_2(t) \, dt \\ &= \int_0^1 \, dt + \int_0^1 \left(4e^{1-t} - e^{4t} \right) \, dt \\ &= 1 + \left(-4e^{1-t} - \frac{1}{4} e^{4t} \right)_0^1 \\ &= 1 - 4 - \frac{1}{4} e^4 + 4e + \frac{1}{4} \\ &= 4e - \frac{1}{4} e^4 - \frac{11}{4}. \end{aligned}$$

5. Let C be the curve from $(0,0,0)$ to $(1,1,1)$ along the intersection of the surfaces $y = x^2$ and $z = x^3$. Find the integral of the vector field $\mathbf{F}(x,y,z) = (x^2, xy, z^2)$ along this curve.

Solution. We first need to parametrize the curve. A point in $\mathbb{R}^3$ on the surface $y = x^2$ has coordinates (t, t^2, z) for $(z,t) \in \mathbb{R}^2$. At the intersection with the surface $z = x^3$, we must also have $z = t^3$. It then follows that points on the intersection of the two surfaces have coordinates (t, t^2, t^3) with $t \in \mathbb{R}$. Now we want our curve to start at $(0,0,0)$ and end at $(1,1,1)$. So our parameter must go from $t = 0$ to $t = 1$. We thus end up with the parametrization

$$\alpha : [0,1] \to \mathbb{R}^3, \qquad \alpha(t) = (t, t^2, t^3).$$

The tangent vector is $\mathbf{T}(t) = (1, 2t, 3t^2)$. The line integral becomes:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\ dt &= \int_0^1 \Big((t^2, t^3, t^6) \cdot (1, 2t, 3t^2) \Big)\ dt \\ &= \int_0^1 \Big(t^2 + 2t^4 + 3t^8 \Big)\ dt \\ &= \frac{1}{3} + \frac{2}{5} + \frac{3}{9} \\ &= \frac{16}{15}.\end{aligned}$$

3.4 Fundamental Theorem of line integrals

In Section 2.2 we studied an important class of one-forms called exact, which arise as differentials of functions. Their associated vector fields are called conservative, and can be expressed as the gradient of a potential function. In this section we see that line integrals of such one-forms are very nice and satisfy beautiful properties. This leads us to the Fundamental Theorem of line integrals, which is a natural generalization of the Fundamental Theorem of Calculus.

Objectives

You should be able to:

- State the Fundamental Theorem of line integrals for line integrals of exact one-forms, and use it to evaluate line integrals.
- Show that the Fundamental Theorem of line integrals implies that line integrals of exact one-forms only depend on the starting and ending points of the curve.
- Show that the Fundamental Theorem of line integrals implies that line integrals of exact one-forms over closed curves vanish.
- State these results in terms of conservative vector fields and their associated potential functions.
- Use the Fundamental Theorem of line integrals and its consequences to show that a given one-form cannot be exact.

3.4.1 The Fundamental Theorem of line integrals

Recall from Section 2.2 that an exact one-form is a one-form that can be written as the differential of a function: $\omega = df$. Conversely, its associated vector field $\mathbf{F}$ can be written as the gradient of a function, $\mathbf{F} = \boldsymbol{\nabla} f$; we say that $\mathbf{F}$ is conservative and that f is its associated potential.

Theorem 3.4.1 The Fundamental Theorem of line integrals. *Let $\omega = df$ be an exact one-form on an open subset $U \subseteq \mathbb{R}^n$, and $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve whose image $\alpha([a, b]) = C \subset U$. Then:*

$$\int_\alpha \omega = \int_\alpha df = f(\alpha(b)) - f(\alpha(a)).$$

The integral thus only depends on the starting and ending points of the image curve C.

Proof. You have probably noticed that this theorem is similar in flavour to the Fundamental Theorem of Calculus for definite integrals; in fact it follows from it, as we will see.

First, by the definition of line integrals, we have:

$$\int_\alpha df = \int_{[a,b]} \alpha^*(df).$$

Next, we can use one of the fundamental properties of the pullback, which is that $\alpha^*(df) = d(\alpha^* f)$. So we can write:

$$\int_\alpha df = \int_{[a,b]} d(\alpha^* f).$$

If we introduce a parameter t for the parametric curve, i.e. $\alpha(t) = (x_1(t), \ldots, x_n(t))$, then $\alpha^* f(t) = f(\alpha(t))$, and we can write the integral as:

$$\int_\alpha df = \int_a^b \frac{d}{dt}(f(\alpha(t))\ dt.$$

But then, the right-hand-side is just a standard definite integral of the derivative of a function. By the Fundamental Theorem of Calculus (part 2), we know that the right-hand-side is simply equal to $f(\alpha(b)) - f(\alpha(a))$. We thus get:

$$\int_\alpha df = f(\alpha(b)) - f(\alpha(a)).$$

■

This result makes it very easy to evaluate line integrals for exact one-forms. But it also has deeper implications. Since the integral only depends on the starting and ending points on the image curve, this means that it does not actually depend on the choice of curve itself! Pick any two parametric curves whose images start and end at the same place: the integral will be the same. This is rather striking!

Corollary 3.4.2 The line integrals of an exact form along two curves that start and end at the same points are equal. *If α and β are two parametric curves whose* *image share the same starting and ending points, and $\omega = df$ is exact, then:*

$$\int_\alpha \omega = \int_\beta \omega.$$

Another direct consequence of the Fundamental Theorem of line integrals is that the integral of an exact one-form over a closed curve always vanishes! Indeed, the curve is closed if $\alpha(b) = \alpha(a)$ (so that the image curve is a "loop"), and so the right-hand-side in Theorem 3.4.1 vanishes.

Corollary 3.4.3 The line integral of an exact one-form along a closed curve vanishes. *Let $\omega = df$ be an exact one-form on an open subset $U \subseteq \mathbb{R}^n$, and α be any closed parametric curve whose image $C \subset U$. Then*

$$\int_\alpha \omega = 0.$$

We sometimes write $\oint_\alpha \omega$ for the line integral of a one-form along a closed parametric curve.

Example 3.4.4 An example of a line integral of an exact one-form. Suppose that you want to integrate the one-form $\omega = y^2 z\ dx + 2xyz\ dy + xy^2\ dz$ over the line segment joining the origin to the point $(1, 1, 1) \in \mathbb{R}^3$. In principle, to evaluate the line integral, you would need to find a parametrization for the line, and use the definition of line integrals Definition 3.3.2 to evaluate the integral. However, we notice here that ω is exact! Indeed, if you pick the function $f(x, y, z) = xy^2 z$, then

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz = y^2 z\ dx + 2xyz\ dy + xy^2\ dz,$$

which is ω. Thus we can use the Fundamental Theorem of line integrals to evaluate the line integral. Let α be any parametrization of the line segment joining the origin to the point $(1, 1, 1)$. We get:

$$\int_\alpha \omega = f(1, 1, 1) - f(0, 0, 0) = 1 - 0 = 1.$$

What is neat as well is that you know that the line integral of ω along *any curve* joining the origin to the point $(1, 1, 1)$ will be equal to 1! The curve does not have to be a line. It could be a parabola, the arc of a circle, anything! For instance, just for fun let us pick the following parametric curve $\beta : [0, 1] \to \mathbb{R}^3$ with $\beta(t) = (t, t^2, t^3)$, whose image curve joins the origin to $(1, 1, 1)$. Let us show that this works. By definition of line integrals,

$$\begin{aligned}
\int_\beta \omega &= \int_0^1 \Big((t^2)^2 t^3 (1) + 2(t)(t^2)(t^3)(2t) + t(t^2)^2(3t^2) \Big)\ dt \\
&= \int_0^1 \Big(t^7 + 4t^7 + 3t^7 \Big)\ dt \\
&= 8 \int_0^1 t^7\ dt \\
&= t^8 \Big|_0^1 \\
&= 1.
\end{aligned}$$

Neat!

One thing that we did not explain however here: how did we know that ω was exact? This is not always so easy to figure out. We will discuss this further in Section 3.6. □

3.4.2 The Fundamental Theorem of line integrals for vector fields

To end this section, let us rewrite the Fundamental Theorem for line integrals in terms of the associated vector fields.

Theorem 3.4.5 The Fundamental Theorem of line integrals for vector fields. *Let $\mathbf{F} = \boldsymbol{\nabla} f$ be a conservative vector field on an open subset $U \subseteq \mathbb{R}^n$, and $\alpha : [a, b] \to \mathbb{R}^n$ a parametric curve whose image $\alpha([a, b]) = C \subset U$. Then:*

$$\int_a^b \mathbf{F}(\alpha(t)) \cdot \mathbf{T}(t)\ dt = \int_a^b \boldsymbol{\nabla} f(\alpha(t)) \cdot \mathbf{T}(t)\ dt = f(\alpha(b)) - f(\alpha(a)).$$

In the notation introduced in Remark 3.3.8, we can rewrite this integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(P_1) - f(P_0),$$

where $P_0 = \mathbf{r}(a)$ and $P_1 = \mathbf{r}(b)$ are the starting and ending points respectively of the curve C with direction of travel specified by the position function $\mathbf{r}$.

Note that from Corollary 3.4.2 and Corollary 3.4.3, we know that:

- the line integral of a conservative vector field does not depend on the path chosen between two points;
- the line integral of a conservative vector field along a closed curve is always zero.

3.4.3 Exercises

1. Consider the one-form $\omega = (y + ze^x)\ dx + (x + e^y \sin z)\ dy + (z + e^x + e^y \cos z)\ dz$ on $\mathbb{R}^3$. Show that ω is an exact form, and use this fact to evaluate the integral of ω along the parametric curve $\alpha : [0, \pi] \to \mathbb{R}^3$ with $\alpha(t) = (t, e^t, \sin t)$.

Solution. To show that it is exact, we simply find a function $f(x, y, z)$ such that $\omega = df$ by inspection (we can also do that by integrating the partial derivatives as we did a number of times already). We guess that $f(x, y, z) = xy + ze^x + e^y \sin z + \frac{1}{2}z^2$, and check that it works. Its differential is:

$$df = (y + ze^x)\ dx + (x + e^y \sin z)\ dy + (z + e^x + e^y \cos z)\ dz,$$

which is indeed ω. So our guess is correct, and we have shown that ω is exact.

Using the Fundamental Theorem of line integrals, we can integrate ω directly along α:

$$\begin{aligned}\int_\alpha \omega =& f(\alpha(\pi)) - f(\alpha(0))\\ =& f(\pi, e^\pi, 0) - f(0, 1, 0)\\ =& \pi e^\pi.\end{aligned}$$

2. Recall from Example 2.2.13 (see also Example 3.6.5) that the one-form $\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is closed. However, we said that it was not exact. Use the integral of ω along one turn counterclockwise around the unit circle to show that ω cannot be exact.

Solution. We parametrize the unit circle as usual by $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with $\alpha(t) = (\cos t, \sin t)$. The pullback one-form is

$$\begin{aligned}\alpha^*\omega =& \left(-\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t) \right)\, dt \\ =&(\sin^2 t + \cos^2 t)\, dt \\ =&dt.\end{aligned}$$

The line integral thus simply becomes:

$$\int_\alpha \omega = \int_0^{2\pi} dt = 2\pi.$$

In particular, it is non-zero. This proves that ω cannot be exact on $\mathbb{R}^2 \setminus \{(0,0)\}$, since if it was exact its line integral along a closed curve would have to vanish.

3. Suppose that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^2$ and that its integral from point $(1, 0)$ to $(-1, 0)$ along the upper half of the unit circle is 5. What should the integral from $(1, 0)$ to $(-1, 0)$ but along the lower half of the circle be?

Solution. It should be 5! Indeed, since $\mathbf{F}$ is conservative, we know that its line integral does not depend on the path chosen between two points. Since both paths here start and end at the same points, the line integrals along these paths must be equal.

4. Consider the one-form $\omega = df$ on $\mathbb{R}^2$ with $f(x, y) = \sin(x + y)$. Find a parametric curve α that is not closed but such that

$$\int_\alpha \omega = 0.$$

Solution. The one-form $\omega = df$ is obviously exact. By the Fundamental Theorem for line integrals, we know that

$$\int_\alpha \omega = f(\alpha(b)) - f(\alpha(a)).$$

If we write $\alpha(t) = (x(t), y(t))$, then this becomes

$$\int_\alpha \omega = \sin(x(b) + y(b)) - \sin(x(a) + y(a)).$$

Now we want this integral to be zero. Thus we want

$$\sin(x(b) + y(b)) = \sin(x(a) + y(a)).$$

But we don't want a closed curve, so we must choose our curve such that $\alpha(b) \neq \alpha(a)$. There are of course many possible choices. Here is one example:

$$\alpha(a) = (0, 0), \qquad \alpha(b) = (\pi, 0).$$

Then
$$\sin(x(a) + y(a)) = \sin(0) = 0, \qquad \sin(x(b) + y(b)) = \sin(\pi) = 0.$$
Thus
$$\int_\alpha \omega = 0$$
for any parametric curve that starts at $(0,0)$ and ends at $(\pi,0)$. For instance, we could pick a straight line between the two points.

5. Let ω be a one-form that is defined on all of $\mathbb{R}^2$. Let P_0, P_1, P_0', P_1' be any four points in $\mathbb{R}^2$. Suppose that
$$\int_{C_1} \omega = \int_{C_2} \omega$$
for any two curves C_1 and C_2 that start at P_0 and end at P_1. Show that it implies that
$$\int_{C_1'} \omega = \int_{C_2'} \omega$$
for any two curves C_1' and C_2' that start at P_0' and end at P_1'.

In other words, if the line integral of a one-form between two given points is path independent, then it is path independent everywhere.

Solution. The proof is fairly intuitive. Fix $P_0, P_1 \in \mathbb{R}^2$, and pick any two other points $P_0', P_1' \in \mathbb{R}^2$. Let D_0 be a fixed curve from P_0 to P_0', and D_1 a fixed curve from P_1' to P_1. Suppose that C_1' and C_2' are two curves from P_0' to P_1'.

On the one hand, the curve $C_1 = D_0 \cup C_1' \cup D_1$ is curve from P_0 to P_1. The line integral of ω along C_1 is
$$\int_{C_1} \omega = \int_{D_0} \omega + \int_{C_1'} \omega + \int_{D_1} \omega.$$
On the other hand, the curve $C_2 = D_0 \cup C_2' \cup D_1$ is also a curve from P_0 to P_1. The line integral of ω along C_2 is
$$\int_{C_2} \omega = \int_{D_0} \omega + \int_{C_2'} \omega + \int_{D_1} \omega.$$
But we know that
$$\int_{C_1} \omega = \int_{C_2} \omega.$$
Equating the two expressions for these line integrals, and simplifying, we end up with the statement that
$$\int_{C_1'} \omega = \int_{C_2'} \omega.$$
Since this must be true for any points P_0' and P_1', and any curves C_1' and C_2' from P_0' to P_1', we conclude that the line integral of ω is path independent everywhere.

3.5 Applications of line integrals

We mentioned in Subsection 3.3.2 that line integrals are sometimes called "work integrals". In this section we explain why, and work through an example where line integrals can be used to calculate the work done. In this section we use vector field notation instead of one-forms, as this is what is most commonly encountered in such applications.

Objectives

You should be able to:

- Determine and evaluate appropriate line integrals in the context of applications in science, in particular for evaluating the work done while moving an object in a force field.

3.5.1 Work

If you pick something off the ground, you expend energy. In physics, this is called "work", because you move an object that is under the influence of a force field. If you move an object along a given path in a force field, how can you find the work done?

The idea is to use the well known "slicing" principle that turns a problem into an integration question. Suppose that there is a force field $\mathbf{F}(x, y, z)$ in $\mathbb{R}^3$, and that we move an object along a path specified by a position vector $\mathbf{r}(t)$ with $a \leq t \leq b$. If the force field was constant and did not depend on the position (x, y, z) of the object, from general physics principles the work done on the object moving along the path would be $\mathbf{F} \cdot (\mathbf{r}(b) - \mathbf{r}(a))$, i.e. the dot product of the force and the displacement (in other words, it is the magnitude of the force times the displacement in the direction of the force). However, as the force field depends on the position of the object, we cannot easily calculate the work directly. But we can slice the problem and sum over slices to rewrite the calculation as an integral.

We slice our time interval $[a, b]$ into small slices of width Δt. Over a small time interval Δt, the object moves from position $\mathbf{r}(t)$ to position $\mathbf{r}(t) + \Delta\mathbf{r}$, where $\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$. If the time interval is small, we can assume that the force is constant, and the work done during this time interval can be calculated, to first order, by $\mathbf{F}(\mathbf{r}(t)) \cdot \Delta\mathbf{r}$. Now we sum over slices, and take the limit where we have an infinite number of infinitesimal time intervals; this turns the calculation into a definite integral:

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

We of course recognize the line integral of a vector field as defined in Lemma 3.3.7. This is why line integrals are called work integrals: if the vector field is a force field, the line integral over a parametrized curve calculates the work done when the objects moves along this curve.

Example 3.5.1 Work done by a (non-conservative) force field. Consider the force field in $\mathbb{R}^2$ given by $\mathbf{F}(x, y) = (y, 5x)$. We will calculate the work done when moving an object along two closed curves:

1. going once around the unit circle counterclockwise, starting and ending at $(1, 0)$;
2. going once around a square counterclockwise, with vertices $(1, 1), (1, -1), (-1, -1), (-1, 1)$,

and starting and ending at $(1, 1)$.

Let us start with the circle (call it C). We parametrize the circle by $\mathbf{r}(t) = (\cos t, \sin t)$, with $0 \leq t \leq 2\pi$. The velocity vector is then $\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t)$. To calculate the work done, we evaluate the line integral along the circle:

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\ dt =& \int_0^{2\pi} ((\sin t)(-\sin t) + (5\cos t)(\cos t))\ dt \\
=& \int_0^{2\pi} \left(-\sin^2 t + 5\cos^2 t\right)\ dt \\
=& \int_0^{2\pi} \left(6\cos^2 t - 1\right)\ dt \\
=& \int_0^{2\pi} (3(1 + \cos(2t)) - 1)\ dt \\
=& \int_0^{2\pi} (2 + 3\cos(2t))\ dt \\
=& 2(2\pi) + \frac{3}{2}\sin(4\pi) - \frac{3}{2}\sin(0) \\
=& 4\pi.
\end{aligned}$$

We see that even if the curve is closed (i.e. it starts and ends at the same point), the work done is non-zero: this is because the force field is not conservative. If it was conservative, by Corollary 3.4.3 the work would have been zero.

Now consider the square (call it S). It is a piecewise parametric curve, so we need to parametrize the four line segments separately. We use the following parametrizations:

- (L_1) From $(1, 1)$ to $(-1, 1)$: $\mathbf{r}_1(t) = (1 - t, 1)$, $0 \leq t \leq 2$;
- (L_2) From $(-1, 1)$ to $(-1, -1)$: $\mathbf{r}_2(t) = (-1, 1 - t)$, $0 \leq t \leq 2$;
- (L_3) From $(-1, -1)$ to $(1, -1)$: $\mathbf{r}_3(t) = (-1 + t, -1)$, $0 \leq t \leq 2$;
- (L_4) From $(1, -1)$ to $(1, 1)$: $\mathbf{r}_4(t) = (1, -1 + t)$, $0 \leq t \leq 2$.

The work done is then calculated by summing the four line integrals:

$$\begin{aligned}
W =& \int_S \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\ dt \\
=& \int_{L_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt}\ dt + \int_{L_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt}\ dt + \int_{L_3} \mathbf{F} \cdot \frac{d\mathbf{r}_3}{dt}\ dt + \int_{L_4} \mathbf{F} \cdot \frac{d\mathbf{r}_4}{dt}\ dt \\
=& \int_0^2 ((1)(-1) + 5(1 - t)(0))\ dt + \int_0^2 ((1 - t)(0) + 5(-1)(-1))\ dt \\
&+ \int_0^2 ((-1)(1) + 5(-1 + t)(0))\ dt + \int_0^2 ((-1 + t)(0) + 5(1)(1))\ dt \\
=& -2 + 10 - 2 + 10 \\
=& 16.
\end{aligned}$$

We see that the work is again non-zero, and in fact it is not the same as the work done when going around the unit circle. This is as expected: as the force is non-conservative, the work done should depend on the path chosen. $\square$

3.5.2 Conservation of energy

A force field that can be written as the gradient of a potential is called "conservative" for a reason. The name comes from physics, as it is related to conservation of energy, as we now see.

Let $\mathbf{F}$ be a force field in $\mathbb{R}^3$, and suppose that an object moves along a parametrized path $\mathbf{r}(t)$ from $t = a$ to $t = b$ (let's call this parametric curve α). By Newton's law, we know that

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2},$$

where m is the mass of the object. From the discussion above, we see that the work done by the force on the object is:

$$\begin{aligned} W =& \int_\alpha \mathbf{F} \cdot d\mathbf{r} \\ =& m \int_a^b \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} \ dt \\ =& m \int_a^b \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right) \ dt - m \int_a^b \frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} \ dt. \end{aligned}$$

We notice that the second integral on the last line is the same as the integral on the previous line, so we end up with the statement that

$$\begin{aligned} m \int_a^b \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} \ dt =& \frac{1}{2} m \int_a^b \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right) \ dt \\ =& \frac{m}{2}|\mathbf{r}'(b)|^2 - \frac{m}{2}|\mathbf{r}'(a)|^2. \end{aligned}$$

Since $\mathbf{r}'(t) = \mathbf{v}(t)$ is the velocity of the object, we recognize the expression $\frac{m}{2}|\mathbf{r}'(t)|^2$ as being the kinetic energy of the object at the point parametrized by t on its path C, which we will denote by $T(t)$. Thus we conclude that the work is the difference in kinetic energy

$$W = T(b) - T(a)$$

between the kinetic energy $T(b)$ of the object at the ending point of the path and its kinetic energy $T(a)$ at the starting point.

Now assume that $\mathbf{F}$ is a conservative force field. It can then be written as the gradient of a potential. We now change our conventions (only for this section), to be consistent with the physics literature, and introduce a minus sign. We write: $\mathbf{F} = -\boldsymbol{\nabla} V$ for some potential function V. By the Fundamental Theorem of line integrals, the work can then be evaluated as follows:

$$\begin{aligned} W =& \int_\alpha \mathbf{F} \cdot d\mathbf{r} \\ =& - \int_a^b \boldsymbol{\nabla} V \cdot d\mathbf{r} \\ =& - (V(\mathbf{r}(b)) - V(\mathbf{r}(a)). \end{aligned}$$

Comparing with our previous calculation of the work, we conclude that

$$T(b) - T(a) = -(V(\mathbf{r}(b)) - V(\mathbf{r}(a))),$$

or, rearranging,

$$T(b) + V(\mathbf{r}(b) = T(a) + V(\mathbf{r}(a)).$$

This statement is the well known law of energy conservation in physics! Indeed, the potential V is interpreted as the potential energy, and the equality says that the total sum of the object's kinetic and potential energies remains constant as the object moves along the path. This is why such forces are called "conservative"!

3.5.3 Exercises

1. The force exerted by an electric charge at the origin on a charged particle at a point $(x, y, z) \in \mathbb{R}^3$ is

$$\mathbf{F}(x, y, z) = \frac{K}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z),$$

where K is a constant. Find the work done as the particle moves along a straight line from $(1, 0, 0)$ to $(2, 2, 3)$.

Solution. We parametrize the line as $\alpha : [0, 1] \to \mathbb{R}^3$ with $\alpha(t) = (1 + t, 2t, 3t)$. The tangent vector is $\mathbf{T}(t) = (1, 2, 3)$. The line integral thus becomes

$$\int_\alpha \mathbf{F} \cdot d\mathbf{r} = K \int_0^1 \frac{1}{((1 + t)^2 + (2t)^2 + (3t)^2)^{3/2}}((1 + t) + 2(2t) + 3(3t))\ dt.$$

We do the substitution $u = (1+t)^2+(2t)^2+(3t)^2$, with $du = 2\left((1 + t) + 2(2t) + 3(3t)\right)\ dt$, and $u(0) = 1,\ u(1) = 4 + 4 + 9 = 17$. The integral becomes

$$\begin{aligned}\int_\alpha \mathbf{F} \cdot d\mathbf{r} =& \frac{K}{2} \int_1^{17} u^{-3/2}\ du \\ =& K\left(1 - \frac{1}{\sqrt{17}}\right).\end{aligned}$$

2. True or False. A force field $\mathbf{F}(x, y, z) = k(x, y, z)$, with k any constant, does no work on a particle that moves once around the unit circle in the xy-plane.

Solution. This is true, since the force field is conservative, and the integral of a conservative vector field around a closed curve is always zero. To show that the force field is conservative, consider the potential $f(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2)$. Then

$$\boldsymbol{\nabla} f = k(x, y, z) = \mathbf{F}.$$

While the above is a sufficient solution, let us compute the line integral for fun, to see that we get zero indeed. We parametrize the circle as $\alpha : [0, 2\pi] \to \mathbb{R}^3$, $\alpha(t) = (\cos t, \sin t, 0)$, with tangent vector $\mathbf{T}(t) = (-\sin t, \cos t, 0)$. The line integral becomes

$$\begin{aligned}\int_\alpha \mathbf{F} \cdot d\mathbf{r} =& k \int_0^{2\pi} (-\sin t \cos t + \sin t \cos t + 0)\ dt \\ =& 0,\end{aligned}$$

as expected.

3. Find the work done by the force field $\mathbf{F}(x, y, z) = (x^2 + y, x + y, 0)$ when moving an object from $(1, 1, 1)$ to $(0, 0, 0)$.

Solution. The question does not specify the path taken between $(1, 1, 1)$ and $(0, 0, 0)$; so we can only calculate the work if the force is conservative, in which case its line integral does not depend on the path chosen.

Fortunately, the force is conservative. Pick the potential $f(x, y, z) = \frac{x^3}{3} + xy + \frac{y^2}{2}$. Then

$$\nabla f = \left(x^2 + y, x + y, 0\right) = \mathbf{F}.$$

Then, using the Fundamental Theorem for line integrals, we calculate the work:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = &\int_C \nabla f \cdot d\mathbf{r}\\ =&f(0, 0, 0) - f(1, 1, 1)\\ =&-\frac{1}{3} - 1 - \frac{1}{2}\\ =&-\frac{11}{6}.\end{aligned}$$

3.6 Poincare's lemma for one-forms

We left one question unanswered: how do we determine whether a one-form is exact? How do we know whether a vector field is conservative? With the Fundamental Theorem of line integrals under our belts, we will be able to answer this question, which is known as Poincare's lemma.

Objectives

You should be able to:

- Show that a one-form defined on all of $\mathbb{R}^2$ or $\mathbb{R}^3$ is exact if and only if it is closed (Poincare's lemma). Understand the extension to simply connected domains.
- Rephrase Poincare's lemma in terms of conservative vector fields.
- Use Poincare's lemma to determine whether a one-form is exact or a vector field conservative.

3.6.1 One-forms defined on all of $\mathbb{R}^n$

Recall the definition of closed one-forms from Definition 2.2.9 and Definition 2.2.14. We know that exact one-forms are necessarily closed, see Lemma 2.2.10 and Lemma 2.2.15. But is the converse statement true? Are closed one-forms necessarily exact? We know that this cannot always be true, as we have already studied an example of a closed one-form that was not exact (see Exercise 3.4.3.2). So when are closed forms necessarily exact?

This is an important question, because showing that a one-form is closed is much easier than showing that it is exact: one only needs to calculate the partial derivatives of its cooordinate functions and show that they satisfy the requirements in Definition 2.2.9 and Definition 2.2.14.)

It turns out that the answer to the question is fairly subtle. There is one simple case however when the statement is always true: it is when the one-form is defined (and smooth, by definition) on all of $\mathbb{R}^n$.

Theorem 3.6.1 Poincare's lemma, version I. *Let ω be a one-form defined on all of $\mathbb{R}^n$. Then ω is exact if and only if ω is closed.*[1]

The corresponding statement in terms of the associated vector field $\mathbf{F}$ is that, if $\mathbf{F}$ is defined and has continuous partial derivatives on all of $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\mathbf{F}$ is conservative if and only if it passes the screening test from Lemma 2.2.11 and Lemma 2.2.16. In the language that we will introduce in Section 4.4, in the case of $\mathbb{R}^3$ one can say that $\mathbf{F}$ is conservative if and only if it is curl-free:

$$\boldsymbol{\nabla} \times \mathbf{F} = 0.$$

Proof. The proof is rather interesting, and in fact constructive, as it provides a way of calculating the function f such that $\omega = df$ if ω is closed. We will write the proof only for $\mathbb{R}^2$, but a similar proof works in $\mathbb{R}^n$.

First, we notice that one direction of implication is clear: we already know from Lemma 2.2.10 that exact one-forms are closed. So all we need to show is the other direction of implication, namely that closed one-forms are exact.

Assume that $\omega = f(x,y)\ dx + g(x,y)\ dy$ is defined on all of $\mathbb{R}^2$, and that it is closed. From Definition 2.2.9, this means that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Let us now construct a function q as follows. We take our one-form ω, and we integrate it along a curve in $\mathbb{R}^2$ which consists of, first, a horizontal line from the origin $(0,0)$ to the point $(x_0, 0)$ for some fixed $x_0 > 0$, and then a vertical line from the point $(x_0, 0)$ to the point (x_0, y_0) for some fixed $y_0 > 0$. This is a piecewise-parametric curve, but we can easily parametrize each line segment. For the line segment from $(0,0)$ to $(x_0, 0)$, we use the parametrization $\alpha_1 : [0, x_0] \to \mathbb{R}^2$ with $\alpha_1(t) = (t, 0)$, and for the second line segment from $(x_0, 0)$ to (x_0, y_0), we use the parametrization $\alpha_2 : [0, y_0] \to \mathbb{R}^2$ with $\alpha_2(t) = (x_0, t)$. The pullbacks of the one-form ω are $\alpha_1^*\omega = f(t,0)\ dt$ and $\alpha_2^*\omega = g(x_0, t)\ dt$. We construct our new function as q as the line integral of ω along this curve:

$$q(x_0, y_0) = \int_{\alpha_1} \omega + \int_{\alpha_2} \omega = \int_0^{x_0} f(t,0)\ dt + \int_0^{y_0} g(x_0, t)\ dt.$$

q is a function of (x_0, y_0). Now we rename the variables (x_0, y_0) to be (x, y), and extend the function to all $(x,y) \in \mathbb{R}^2$, not just positive numbers, as the integrals on the right-hand-side remain well defined. So we get the function

$$q(x,y) = \int_0^x f(t,0)\ dt + \int_0^y g(x,t)\ dt$$

defined on $\mathbb{R}^2$.

Our claim is that this new function $q(x,y)$ is in fact the potential function for ω, i.e., $\omega = dq$, which would of course show that ω is exact. So let us compute dq. To do so, we need $\frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$. First,

$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y} \int_0^x f(t,0)\ dt + \frac{\partial}{\partial y} \int_0^y g(x,t)\ dt$$

$$=g(x,y)$$

where we used the Fundamental Theorem of Calculus part 1 for the second integral (recalling that x is kept fixed when we evaluate the partial derivative with respect to y) and the fact that the first integral does not depend on y at all. As for the partial derivative with respect to x, we get:

$$\begin{aligned}
\frac{\partial q}{\partial x} =& \frac{\partial}{\partial x}\int_0^x f(t,0)\ dt + \frac{\partial}{\partial x}\int_0^y g(x,t)\ dt \\
=& f(x,0) + \int_0^y \frac{\partial g(x,t)}{\partial x}\ dt \qquad \text{by FTC Part 1 for the first term} \\
=& f(x,0) + \int_0^y \frac{\partial f(x,t)}{\partial t}\ dt \qquad \text{since } \frac{\partial g(x,t)}{\partial x} = \frac{\partial f(x,t)}{\partial t}, \text{ as } \omega \text{ is closed,} \\
=& f(x,0) + f(x,y) - f(x,0) \\
=& f(x,y).
\end{aligned}$$

Therefore,

$$dq = \frac{\partial q}{\partial x}\ dx + \frac{\partial q}{\partial y}\ dy = f(x,y)\ dx + g(x,y)\ dy = \omega,$$

and we have shown that ω is exact. Moreover, we found an explicit expression for the potential function as a line integral of ω. ■

So we now have a clear criterion to determine whether a one-form on $\mathbb{R}^n$ is exact: we simply need to show that is closed. In terms of vector fields, all we have to do is show that it passes the screening test.

Example 3.6.2 Closed forms are exact. Consider the example from Example 3.4.4. The one-form was $\omega = y^2 z\ dx + 2xyz\ dy + xy^2\ dz$, and we know that it is exact, as it can be written as $\omega = df$ for $f(x,y,z) = xy^2 z$. But suppose that we don't know that. How can we determine quickly whether it is exact or not?

First, we notice that ω is well defined on all of $\mathbb{R}^3$. So to determine that it is exact, all that we need to do is show that it is closed.

Let us write $\omega = f_1\ dx + f_2\ dy + f_3\ dz$. We calculate partial derivatives:

$$\frac{\partial f_1}{\partial y} = 2yz, \qquad \frac{\partial f_1}{\partial z} = y^2, \qquad \frac{\partial f_2}{\partial z} = 2xy,$$

and

$$\frac{\partial f_2}{\partial x} = 2yz, \qquad \frac{\partial f_3}{\partial x} = y^2, \qquad \frac{\partial f_3}{\partial y} = 2xy.$$

The statement that ω is closed is just that the partial derivatives in the first line are equal to the partial derivatives in the second line, which is indeed true. Thus ω is closed, and by Poincare's lemma we can conclude that it must be exact.

That doesn't tell us how to find the potential function f though. To find f, we proceed as usual. Let us do it here for completeness.

[1]To be precise, we only defined closeness for one-forms in $\mathbb{R}^2$ and $\mathbb{R}^3$ at this stage. But the statement holds in $\mathbb{R}^n$ using the general definition of closed one-forms in Definition 4.6.1 -- see Theorem 4.6.4.

We want to find a function $f = f(x,y,z)$ such that $df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz = y^2 z\ dx + 2xyz\ dy + xy^2\ dz$. First, we want:

$$\frac{\partial f}{\partial x} = y^2 z.$$

We can integrate the partial derivative -- the "constant of integration" here will be any function $g(y,z)$ that is independent of x. Thus we get:

$$f = \int_0^x y^2 z\ dx + g(y,z) = xy^2 z + g(y,z).$$

Next, we want

$$\frac{\partial f}{\partial y} = 2xyz.$$

Using the fact that $f = xy^2 z + g(y,z)$, this equation reads

$$2xyz + \frac{\partial g}{\partial y} = 2xyz \qquad \Leftrightarrow \qquad \frac{\partial g}{\partial y} = 0.$$

Integrating, we get:

$$g(y,z) = h(z),$$

where $h(z)$ is a function of z alone. Putting this together, we have $f = xy^2 z + h(z)$. Finally, we must satisfy the remaining equation:

$$\frac{\partial f}{\partial z} = xy^2.$$

Using the fact that $f = xy^2 z + h(z)$, this becomes

$$xy^2 + \frac{dh}{dz} = xy^2 \qquad \Leftrightarrow \qquad \frac{dh}{dz} = 0 \qquad \Leftrightarrow \qquad h = C,$$

for some constant C. As we are only interested in one function f such that $df = \omega$, we can set the constant $C = 0$. We obtain that $\omega = df$ with $f(x,y,z) = xy^2 z$, as stated in Example 3.4.4. □

In fact, we can go a little further and state the following theorem, which gives equivalent formulations of what it means for a one-form to be exact (or a vector field to be conservative) on all of $\mathbb{R}^n$.

Theorem 3.6.3 Equivalent formulations of exactness on $\mathbb{R}^n$. *Let ω be a one-form defined on all of $\mathbb{R}^n$, and $\mathbf{F}$ its associated vector field. Then the following statements are equivalent:*

1. *ω is exact ($\mathbf{F}$ is conservative).*
2. *ω is closed ($\mathbf{F}$ passes the screening test).*
3. *The line integral $\int_\alpha \omega = 0$ for any closed parametric curve α.*
4. *Line integrals of ω are path independent.*

In other words, if one of these statements is true, then all four statements are true.

Proof. We want to prove equivalence of the four statements. To do so, it is sufficient to prove that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1). We will write a proof only for $\mathbb{R}^2$.

(1) $\Rightarrow$ (2). All exact one-forms are closed, see Lemma 2.2.10.

(2) $\Rightarrow$ (3). By Poincare's lemma, Theorem 3.6.1, we know that closed one-forms defined on all of $\mathbb{R}^n$ are exact, so (1) $\Leftrightarrow$ (2). We also know that if ω is exact, then its line integral along closed curves always vanishes: this is Corollary 3.4.3, which follows from the Fundamental Theorem of line integrals. So (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4). This follows from Exercise 3.4.3.5. Indeed, suppose that P_0 is on the closed curve that you are integrating along. Pick $P_1 = P_0$. Then we know that the line integral of ω is path independent for all curves starting at P_0 and ending at $P_1 = P_0$, since by (3) the line integrals all vanish. It then follows from Exercise 3.4.3.5 that the line integrals are path independent everywhere, which is (4).

(4) $\Rightarrow$ (1). For this one we need to do a bit more work. We want to show that if the line integrals of $\omega = f\ dx + g\ dy$ are path independent, then ω is exact. We proceed like in the proof of Theorem 3.6.1. First, consider a curve C_1 which consists in a horizontal line from $(0,0)$ to a fixed point $(x_0, 0)$, and then a vertical line from $(x_0, 0)$ to a fixed point (x_0, y_0), with $x_0, y_0 > 0$. We parametrize it by $\alpha_1 : [0, x_0] \to \mathbb{R}^2$ with $\alpha_1(t) = (t, 0)$, and $\alpha_2 : [0, y_0] \to \mathbb{R}^2$ with $\alpha_2(t) = (x_0, t)$. The pullbacks are $\alpha_1^*\omega = f(t, 0)\ dt$, $\alpha_2^*\omega = g(x_0, t)\ dt$. The line integral then reads

$$q(x_0, y_0) := \int_{C_1} \omega = \int_0^{x_0} f(t, 0)\ dt + \int_0^{y_0} g(x_0, t)\ dt.$$

Next, we consider a second curve C_2 which consists in a vertical line from $(0,0)$ to $(0, y_0)$, and then a horizontal line from $(0, y_0)$ to (x_0, y_0). A parametrization is $\beta_1 : [0, y_0] \to \mathbb{R}^2$ with $\beta_1(t) = (0, t)$, and $\beta_2 : [0, x_0] \to \mathbb{R}^2$ with $\beta_2(t) = (t, y_0)$. The pullbacks are $\beta_1^*\omega = g(0, t)\ dt$, and $\beta_2^*\omega = f(t, y_0)\ dt$. The line integral reads

$$p(x_0, y_0) := \int_{C_2} \omega = \int_0^{y_0} g(0, t)\ dt + \int_0^{x_0} f(t, y)\ dt.$$

Note that the two curves C_1 and C_2 start at $(0,0)$ and end at the same point (x_0, y_0). Since by (4) we know that the line integrals are path independent, we know that

$$q(x_0, y_0) = p(x_0, y_0).$$

As in the proof of Theorem 3.6.1, we then rename the variables $(x_0, y_0) \to (x, y)$ and extend the domain of definition of the function $q(x, y) = p(x, y)$ to all $(x, y) \in \mathbb{R}^2$, since the integrals on the right-hand-side remain well defined.

Since $q(x, y) = p(x, y)$, the partial derivatives of q and p are also equal. In particular,

$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y} \int_0^y g(x, t)\ dt = g(x, y),$$

and

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \int_0^x f(t, y)\ dt = f(x, y),$$

where in both cases we used FTC part 1. We conclude that

$$dq = \frac{\partial q}{\partial x}\ dx + \frac{\partial q}{\partial y}\ dy = f(x, y)\ dx + g(x, y)\ dy = \omega,$$

and hence ω is exact, which is (1). ■

3.6.2 One-forms on simply connected subsets of $\mathbb{R}^n$

Going back to Poincare's lemma, the proof of Theorem 3.6.1 relied on the fact that we could take x and y to be any two real numbers, which was possible because ω was assumed to be a (smooth) one-form on all of $\mathbb{R}^2$. But if the one-form is defined on an open subset $U \subseteq \mathbb{R}^2$, does the proof still work? The answer is: not always. For instance, it would work if U is an open rectangle in $\mathbb{R}^2$; however, it wouldn't work if U is $\mathbb{R}^2 \setminus \{(0,0)\}$, that is $\mathbb{R}^2$ minus the origin.

In other words, it isn't always true that closed one-forms are exact. The precise statement is that it is true if ω is a one-form defined on an open subset $U \subseteq \mathbb{R}^n$ that is "simply connected". What does this mean?

We say that a set U is **path connected** if any two points can be connected by a path (or a parametric curve). In other words, the set contains only one piece. Then, we say that it is **simply connected** if it is path connected, with the extra property that any simple closed curve (loop) in U can be continuously contracted to a point. Intuitively, a simply connected region in $\mathbb{R}^2$ consists of only one piece and has no holes.

For instance, open rectangles and open disks in $\mathbb{R}^2$ are simply connected. However, if you consider $U = \mathbb{R}^2 \setminus \{(0,0)\}$, while it is path connected as you can connect any two points by a path, it is not simply connected, since loops around the origin cannot be contracted to a point within U (there is a hole at the origin).

We will state the more general Poincare's lemma here for completeness, but without a proof, as it would go beyond the scope of this course.

Theorem 3.6.4 Poincare's lemma, version II. *Let ω be a one-form defined on an open subset $U \subseteq \mathbb{R}^n$ that is simply connected. Then ω is exact if and only if it is closed.*

Example 3.6.5 An example of a closed one-form that is not exact. We saw in Example 2.2.13 an example of a one-form that is closed but not exact. The one-form was

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

We showed that it is closed. But we also showed in Exercise 3.4.3.2 that it is not exact, since its line integral around a closed curve is non-vanishing. Does that contradict Poincare's lemma? No. The reason is that ω is not defined on all of $\mathbb{R}^2$. Indeed, its coefficient functions are only defined for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, as at the origin one would be dividing by zero. As $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected, Poincare's lemma does not apply. □

3.6.3 Exercises

1. Determine whether the one-form $\omega = y^2 e^{xy}\,dx + (1+xy)e^{xy}\,dy$ is exact. If it is, find a function f such that $\omega = df$.

Solution. First, we notice that the component functions are smooth on $\mathbb{R}^2$, so we know that ω is exact if and only if it is closed. We calculate the partial derivatives of the component functions:

$$\frac{\partial}{\partial y}(y^2 e^{xy}) = 2ye^{xy} + xy^2 e^{xy}, \qquad \frac{\partial}{\partial x}\left((1+xy)e^{xy}\right) = ye^{xy} + y(1+xy)e^{xy}.$$

We see that the two expressions are equal. Thus ω is closed, and hence it is also exact.

We are looking for a function f such that

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = y^2 e^{xy}\, dx + (1+xy)e^{xy}\, dy.$$

Integrating the partial derivative in x, we get

$$f = ye^{xy} + g(y).$$

Substituting in the partial derivative for y, we get

$$e^{xy} + xye^{xy} + g'(y) = (1+xy)e^{xy},$$

from which we conclude that $g'(y) = 0$, i.e. $g(y) = C$, which we set to zero. We thus have found a function f such that $\omega = df$:

$$f(x,y) = ye^{xy}.$$

2. Determine whether the field $\mathbf{F}(x,y) = (e^x + e^y, xe^y + x)$ is conservative. If it is, find a potential function.

Solution. The component functions are smooth on $\mathbb{R}^2$, so the vector field is conservative if and only if it passes the screening test. We calculate the partial derivatives:

$$\frac{\partial}{\partial y}(e^x + e^y) = e^y, \qquad \frac{\partial}{\partial x}(xe^y + x) = e^y + 1.$$

As these two expressions are not equal, we conclude that the vector field is not conservative on $\mathbb{R}^2$.

3. Determine whether or not the following sets are (a) open, (b) path connected, and (c) simply connected:

(a) $S = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$

(b) $U = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (1,1)\}$

(c) $T = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$

(d) The unit circle in $\mathbb{R}^2$

(e) The unit sphere in $\mathbb{R}^3$

(f) $V = \mathbb{R}^3 \setminus \{(0,0,0)\}$

Solution. Recall that a set is open if for all points in the set, there is an open ball centered at that point that lies within the set. It is path connected if any two points in the set can be connected by a path. It is simply connected if it is path connected, and all closed curves can be contracted to a point within the set.

(a) S consists in the upper half of the xy-plane, including the x-axis. First, it is not open, since any point on the x-axis cannot be the centre of an open disk within S (as points below the x-axis are not in S). It is however path connected, as any two points can be connected by a path, and it is simply connected, as all closed curves can be contracted to a point within S.

(b) U is the xy-plane minus the point $(1, 1)$. It is certainly open and path connected, but it is not simply connected as any closed curve surrounding $(1, 1)$ cannot be contracted to a point in U (as there is a hole at $(1, 1)$).

(c) T is the xy-plane with the x-axis removed. It is an open set. However, it is not path connected, since two points on both sides of the x-axis cannot be connected by a path within T. It then follows that it is also not simply connected.

(d) The unit circle in $\mathbb{R}^2$, i.e. the solutions to the equation $x^2 + y^2 = 1$, is not open in $\mathbb{R}^2$. Indeed, there is no point on the unit circle that can be surrounded by an open disk within the circle itself (note that we are considering only the circle itself here, not the disk). It is path connected, as you can connect any two points on the circle by a path on the circle (the arc between the two points). But it is not simply connected, as the circle itself (which is a closed loop) cannot be contracted to a point (recall that the interior of the circle is not part of our set).

(e) The unit sphere in $\mathbb{R}^3$ is not open, just as for the circle in $\mathbb{R}^2$. It is path connected, and in this case it is also simply connected, as all closed loops on the sphere can be contracted to a point within the sphere.

(f) V is $\mathbb{R}^3$ minus the origin. It is certainly open, as we are just removing one point from $\mathbb{R}^3$, and it is path connected, as any two points can be connected by a path within the set V. Is it also simply connected? The answer is yes, it is simply connected. Indeed, pick any closed curve within V; you can always contract it to a point in V. The hole at the origin does not create any issue here, because we are in $\mathbb{R}^3$; informally, the point is that even if you pick, say, a closed curve in the xy-plane that surrounds the origin, then you can still contract it to a point within the set V, since you can move up in the z-direction while you contract (you don't have to stick to the xy-plane in the contraction process, as V is a subset in $\mathbb{R}^3$). The upshot here is that it's important to keep in mind that the interpretation of simply-connectedness as meaning "no holes" is only true in $\mathbb{R}^2$. For instance, in $\mathbb{R}^3$, you can convince yourself that the requirement that all closed curves can be contracted to a point within the set could instead be interpreted as meaning that there are no "missing lines" in the set. Ultimately, it is easier to just use the definition of simply connectedness, i.e. that the set is path connected and that all closed curves can be contracted to a point within the set, to check whether a set is simply connected.

4. Consider the one-form $\omega = 2\frac{x}{|y|}\ dx - \frac{x^2}{y^2}\ dy$ on the open subset $U = \{(x,y) \in \mathbb{R}^2 \mid y < 0\}$. Determine whether ω is exact, and if it is find a function f such that $\omega = df$.

Solution. Since we are restricting to $y < 0$, we can replace $|y| = -y$. The one-form is then

$$\omega = -2\frac{x}{y}\ dx - \frac{x^2}{y^2}\ dy.$$

Since U is simply connected, Poincare's lemma applies. We calculate partial derivatives:

$$\frac{\partial}{\partial y}\left(-2\frac{x}{y}\right) = 2\frac{x}{y^2}, \qquad \frac{\partial}{\partial x}\left(-\frac{x^2}{y^2}\right) = -2\frac{x}{y^2}.$$

As the two expressions are not equal, we conclude that ω is not closed on U, and thus it cannot be exact.

We remark here that the choice of U was very important. If we had instead defined the one-form on the open subset $V = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, then we would have obtained a different result! Indeed, on V, $|y| = y$. It then follows that ω is closed, and since Poincare's lemma applies as V is simply connected, it is also exact on V. Indeed, one can check that $\omega = df$ with $f(x,y) = \frac{x^2}{y}$ on V.

In fact, the largest domain of definition for ω would be $U \cup V$, i.e. the set $\{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$. However, this is not a simply connected set, so we cannot apply Poincare's lemma. In any case, ω is not exact on this larger set, as it cannot be written as $\omega = df$ on all of $U \cup V$.

Chapter 4

k-forms

In this section we go beyond one-forms and introduce the general theory of differential forms, with a focus on $\mathbb{R}^3$.

4.1 Differential forms revisited: an algebraic approach

Now that we are familiar with one-forms, we take a step back, and revisit the definition. We introduce a more algebraic approach to one-forms, which will allow us to generalize it and introduce the concept of k-forms.

Objectives

You should be able to:

- Define basic one-forms as linear maps, and basic two- and three-forms as multilinear maps.
- Define k-forms in $\mathbb{R}^3$.
- Prove the antisymmetry propeties of basic k-forms.
- Relate two-forms to vector fields.

4.1.1 An algebraic approach to one-forms

When we introduced one-forms in Definition 2.1.1, we said that the objects dx, dy and dz could be understood a placeholders. We paused briefly in Remark 2.2.4 to relate these objects to the differentials of "projection functions" on $\mathbb{R}^3$, but this wasn't entirely satisfactory. Now we go back and give a more rigorous definition of what these objects stand for.

Definition 4.1.1 The basic one-forms. The **basic one-form** dx_i, with $i \in \{1, \ldots, n\}$, is a linear map $dx_i : \mathbb{R}^n \to \mathbb{R}$ which takes a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and projects it onto the x_i-axis:

$$dx_i(u_1, \ldots, u_n) = u_i.$$

◊

Remark 4.1.2 When we are working on $\mathbb{R}^3$, it is customary to write (x, y, z) for (x_1, x_2, x_3), and we write the basic one-forms as dx, dy, and dz (instead of dx_1, dx_2 and dx_3).

This gives a rigorous meaning of these placeholders. Using this definition, we can write a general linear map $M : \mathbb{R}^3 \to \mathbb{R}$ as

$$M = A\ dx + B\ dy + C\ dz,$$

where $A, B, C \in \mathbb{R}$ are just constants. In other words, it is an arbitrary linear combination of the three projection operators. In general, given an abstract vector space V, the set of linear maps $M : V \to \mathbb{R}$ forms a vector space itself, which is called the "dual vector space" and denoted by V^* (see for instance `https://en.wikipedia.org/wiki/Dual_space`).

Let us now look back at the definition of one-forms in Definition 2.1.1, focusing on $\mathbb{R}^3$ for simplicity. A (differential) one-form was defined as a linear combination of basic one-forms with coefficients that are smooth functions on an open subset $U \subseteq \mathbb{R}^3$. That is, we can write a one-form as:

$$\omega = f\ dx + g\ dy + h\ dz,$$

for smooth functions $f, g, h : U \to \mathbb{R}$. With our new understanding of the placeholders dx, dy, dz, we can make sense of this object. For any point $P \in U$, the one-form ω defines a linear map $\mathbb{R}^3 \to \mathbb{R}$ (or equivalently an element of the vector space dual to $\mathbb{R}^3$). This is the dual concept to vector fields: a vector field is a rule that assigns to all points on U a vector in $\mathbb{R}^3$, while a one-form is a rule that assigns to all points on U a linear map $\mathbb{R}^3 \to \mathbb{R}$ (that is, a "dual vector"). Nice!

4.1.2 Basic k-forms

This algebraic understanding of the basic one-forms as linear maps allows us to define a natural generalization. Instead of looking only at linear maps $\mathbb{R}^n \to \mathbb{R}$, we now also define multilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and so on. The definition relies on the determinant.

Definition 4.1.3 Basic two-forms. The **basic two-form** $dx_i \wedge dx_j$, for $i, j \in \{1, \ldots, n\}$ is a multilinear map[1] $dx_i \wedge dx_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which takes two vectors $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, with $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, and maps them to the following determinant:

$$dx_i \wedge dx_j(\mathbf{u}, \mathbf{v}) = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix}.$$

◊

In the same way we can define the notion of basic three-forms.

Definition 4.1.4 Basic three-forms. The **basic three-form** $dx_i \wedge dx_j \wedge dx_k$, for $i, j, k \in \{1, \ldots, n\}$ is a multilinear map $dx_i \wedge dx_j \wedge dx_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which takes three vectors $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, with $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, and

[1] A **multilinear map** is a function of several variables that is linear separately in each variable.

$\mathbf{w} = (w_1, \ldots, w_n)$, and maps them to the following determinant:

$$dx_i \wedge dx_j \wedge dx_k(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{pmatrix} u_i & v_i & w_i \\ u_j & v_j & w_j \\ u_k & v_k & w_k \end{pmatrix}.$$

$\Diamond$

We wrote the definition of basic one-, two-, and three-forms explicitly for clarity, but in fact they are just special cases of the more general definition of basic k-forms. We now define basic k-forms, for completeness, but don't worry if the notation is confusing you: as we will see, in $\mathbb{R}^3$ all basic k-forms with $k \geq 4$ automatically vanish, so the above definitions are sufficient.

Definition 4.1.5 Basic k-forms. The **basic k-form** $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, with $i_1, \ldots, i_k \in \{1, \ldots, n\}$, is a multilinear map $dx_{i_1} \wedge \ldots \wedge dx_{i_k} : (\mathbb{R}^n)^k \to \mathbb{R}$ which takes k vectors $(\mathbf{u}^1, \ldots, \mathbf{u}^k) \in (\mathbb{R}^n)^k$, with $\mathbf{u}^j = (u_1^j, \ldots, u_n^j)$, and maps them to the following determinant:

$$dx_{i_1} \wedge \ldots \wedge dx_{i_k}(\mathbf{u}^1, \ldots, \mathbf{u}^k) = \det \begin{pmatrix} u_{i_1}^1 & \ldots & u_{i_1}^k \\ \vdots & \ddots & \vdots \\ u_{i_k}^1 & \ldots & u_{i_k}^k \end{pmatrix}.$$

$\Diamond$

It looks like there are many possibilities here, but many of them either vanish or are related to each other. More precisely, basic k-forms satisfy the following antisymmetry relations, which significantly reduce the number of non-zero basic forms $\mathbb{R}^3$.

Lemma 4.1.6 Antisymmetry of basic k-forms. *The basic two-forms satisfy the following properties. For any $i, j \in \{1, \ldots, n\}$,*

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

In particular,

$$dx_i \wedge dx_i = 0, \qquad i \in \{1, \ldots, n\}.$$

Similarly, the basic k-forms pick a sign whenever we exchange the order of two of the dx_i's, and vanish if two of the dx_i's are the same.

It follows that the only non-vanishing basic k-forms in $\mathbb{R}^n$ are for $1 \leq k \leq n$. In particular, in $\mathbb{R}^3$ we only have basic one-, two-, and three-forms. The independent basic two-forms in $\mathbb{R}^3$ are (using the x, y, z notation for $\mathbb{R}^3$):

$$dy \wedge dz, \qquad dz \wedge dx, \qquad dx \wedge dy,$$

and the only independent basic three-form in $\mathbb{R}^3$ is

$$dx \wedge dy \wedge dz.$$

Proof. This follows directly from the property of the determinant. If we exchange a dx_i with dx_j, we exchange two rows in the matrix that we take the determinant of, and hence the determinant picks a sign. Similarly, if two of the dx_i's are the same, then the matrix that we take the determinant of has two equal rows, and hence the determinant is zero. ∎

Remark 4.1.7 We note here that the basic k-forms can be given a geometric interpretation. We focus on $\mathbb{R}^3$ for simplicity. It's easier to start with the basic three-form in $\mathbb{R}^3$. The three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ span a three-dimensional parallelepiped. Then the basic three-form $dx \wedge dy \wedge dz(\mathbf{u}, \mathbf{v}, \mathbf{w})$ calculates its oriented volume, since this is what the determinant calculates.

As for the basic two-forms, the two vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ span a parallelogram. The basic two-form $dx \wedge dy(\mathbf{u}, \mathbf{v})$ calculates the oriented area of its projection on the xy-plane, while the two-forms $dx \wedge dz(\mathbf{u}, \mathbf{v})$ and $dy \wedge dz(\mathbf{u}, \mathbf{v})$ calculate the oriented area of its projection on the xz- and yz-planes respectively.

4.1.3 k-forms in $\mathbb{R}^3$

We are now ready to introduce the concept of k-forms, with $k \in \{0, 1, 2, 3\}$, in $\mathbb{R}^3$, which naturally generalizes the one-forms introduced in Definition 2.1.1.

Definition 4.1.8 k-forms in $\mathbb{R}^3$. Let $U \subseteq \mathbb{R}^3$ be an open subset.

1. A **zero-form** is a smooth function $f : U \to \mathbb{R}$.

2. A **one-form** is an expression of the form

$$f\ dx + g\ dy + h\ dz,$$

   for smooth functions $f, g, h : U \to \mathbb{R}$.

3. A **two-form** is an expression of the form

$$f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy,$$

   for smooth functions $f, g, h : U \to \mathbb{R}$.

4. A **three-form** is an expression of the form

$$f\ dx \wedge dy \wedge dz,$$

   for a smooth function $f : U \to \mathbb{R}$.

◊

Of course there is no point in defining k-forms with $k \geq 4$ in $\mathbb{R}^3$, as those would necessarily vanish, by Lemma 4.1.6. But we note that this definition of k-forms can naturally be generalized to $\mathbb{R}^n$ using the general definition of basic k-forms in Definition 4.1.5.

Remark 4.1.9 Our definition of two- and three-forms involves a specific choice of basic two- and three-forms. For instance, we used $dx \wedge dy \wedge dz$ instead of $dz \wedge dx \wedge dy$. When we express the differential forms with the choice of basic form in Definition 4.1.8, we say that the k-forms are in **standard form**. We generally want to present differential forms in standard form, to simplify things.

With this being said, you may wonder why we chose this particular choice of ordering for the basic two-forms, i.e. why we wrote

$$f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$$

instead of, say,

$$f\ dx \wedge dy + g\ dx \wedge dz + h\ dy \wedge dz.$$

The reason behind this choice will become clear in the next section, when we relate the wedge product of differential forms to the cross-product of vector fields.

An easy way to remember this choice of ordering is to rename in your head $(x, y, z) \mapsto (x_1, x_2, x_3)$, and to denote the component functions by f_1, f_2, f_3. Then the proper choice of ordering is

$$f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2,$$

which runs through the three cyclic permutations of $(1, 2, 3)$), namely $(1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$.

Remark 4.1.10 Using the algebraic interpretation of the basic one-, two-, and three-forms in the previous subsections, we can give a geometric meaning to k-forms: a 0-form assigns a number to all points in U, while a k-form (with $k \geq 1$) assigns a multilinear map $(\mathbb{R}^3)^k \to \mathbb{R}$ to all points in U. In other words, if we act on a given set of vectors, a k-form assigns a notion of k-dimensional oriented volume for the corresponding projection of the k-dimensional parallepiped generated by the k vectors.

4.1.4 k-forms and vector calculus

As has become customary, we end this section by relating our construction in the world of differential forms to the traditional concepts in vector calculus. In Principle 2.1.3, we saw that we can naturally associate to a one-form a corresponding vector field. This correspondence can be generalized to k-forms in $\mathbb{R}^3$. We get the following table, which provides a dictionary between differentials forms in $\mathbb{R}^3$ and vector calculus concepts. Note that to establish the dictionary, we write the k-forms on the left-hand-side of the table in standard form (see Remark 4.1.9).

Table 4.1.11 Dictionary between k-forms in $\mathbb{R}^3$ and vector calculus concepts

Differential form concept		Vector calculus concept	
0-form	f	function	f
1-form	$f\ dx + g\ dy + h\ dz$	vector field	(f, g, h)
2-form	$f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$	vector field	(f, g, h)
3-form	$f\ dx \wedge dy \wedge dz$	function	f

4.1.5 Exercises

1. Show that $dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$.

Solution. We know that $dz \wedge dx = -dx \wedge dz$, and $dz \wedge dy = -dy \wedge dz$. So we can write

$$dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy = dx \wedge dy \wedge dz.$$

2. List the independent non-vanishing basic k-forms in $\mathbb{R}^4$.

Solution. Because of anti-symmetry, we know that the only non-vanishing basic k-forms in $\mathbb{R}^4$ are for $k \leq 4$. Let us write the basic one-forms by

$$dx_1, \qquad dx_2, \qquad dx_3, \qquad dx_4.$$

Then the basic two-forms are obtained by pairing those two-by-two, up to anti-symmetry. We get the basic two-forms:

$$dx_1 \wedge dx_2, \qquad dx_1 \wedge dx_3, \qquad dx_1 \wedge dx_4, \qquad dx_2 \wedge dx_3, \qquad dx_2 \wedge dx_4, \qquad dx_3 \wedge dx_4.$$

For the basic three-forms, we pair the one-forms three-by-three, without repeated factors (otherwise they would vanish). We get:

$$dx_1 \wedge dx_2 \wedge dx_3, \qquad dx_1 \wedge dx_2 \wedge dx_4, \qquad dx_1 \wedge dx_3 \wedge dx_4, \qquad dx_2 \wedge dx_3 \wedge dx_4.$$

Finally, there is only one independent basic four-form, since there cannot be repeated dx_i factors. (There is always only one independent basic "top-form", i.e. basic n-form in $\mathbb{R}^n$). It reads:

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

3. Write down the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ associated to the two-form

$$\omega = xy\ dx \wedge dy + xyz\ dx \wedge dz + x^2 dy \wedge dz.$$

Solution. Before we extract the vector field we need to make sure that we write the one-form in the correct form according to the dictionary Table 4.1.11. We have:

$$\omega = x^2\ dy \wedge dz - xyz\ dz \wedge dx + xy\ dx \wedge dy.$$

Then the associated vector field is:

$$\mathbf{F}(x, y, z) = (x^2, -xyz, xy).$$

4. Let $\omega = dx \wedge dz$ be a basic two-form on $\mathbb{R}^3$, and $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (3, 2, 1)$ be vectors in $\mathbb{R}^3$. Evaluate

$$\omega(\mathbf{u}, \mathbf{v}).$$

Solution. By definition of a basic two-form (and recalling that we use the standard notation here that $dx \wedge dz = dx_1 \wedge dx_3$), we have:

$$dx \wedge dz(\mathbf{u}, \mathbf{v}) = \det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix}.$$

Substituting the entries for the vectors $\mathbf{u}$ and $\mathbf{v}$, we get:

$$dx \wedge dz(\mathbf{u}, \mathbf{v}) = \det \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = 1 - 9 = -8.$$

5. Let $\omega = dx \wedge dy \wedge dz$ be the basic three-form on $\mathbb{R}^3$.

(a) Show that $\omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 1$, with $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$ and $\mathbf{w} = (0, 0, 1)$ basis vectors in $\mathbb{R}^3$.

(b) Show that $\omega(\mathbf{v}, \mathbf{u}, \mathbf{w}) = -1$.

(c) In general, show that there are three choices of ordering of the basis vectors for which ω evaluates to 1, and three choices for which it evaluates to -1.

Solution. (a) By definition of a basic three-form, we get:

$$dx \wedge dy \wedge dz(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

(b) We changed the ordering of the basis vectors here. We get:

$$dx \wedge dy \wedge dz(\mathbf{v}, \mathbf{u}, \mathbf{w}) = \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

(c) In general, it is easy to see that the following three orderings give 1:

$$dx \wedge dy \wedge dz(\mathbf{u}, \mathbf{v}, \mathbf{w}) = dx \wedge dy \wedge dz(\mathbf{w}, \mathbf{u}, \mathbf{v}) = dx \wedge dy \wedge dz(\mathbf{v}, \mathbf{w}, \mathbf{u}) = 1,$$

while the following three orderings give -1:

$$dx \wedge dy \wedge dz(\mathbf{v}, \mathbf{u}, \mathbf{w}) = dx \wedge dy \wedge dz(\mathbf{w}, \mathbf{v}, \mathbf{u}) = dx \wedge dy \wedge dz(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -1.$$

The reason is that whenever we permute two basis vectors, we exchange two columns in the matrix that we are taking the determinant of. But we know from properties of the determinant that swapping two columns of matrix changes its detereminant by -1. So we conclude that doing an even number of two-by-two swaps of basis vectors does not change the determinant, while doing an odd numbers of two-by-two swaps changes the determinant by -1.

FYI: in the language of group theory, the group of permutations of three objects is called the "symmetric group" S_3, whose elements are the permutations. We call a permutation that is a swap of two objects a "transposition". All permutations can be obtained by doing a finite number of transpositions (i.e. by swapping objects two-by-two a finite number of time). We define the "sign" of a permutation as being positive if the permutation can be obtained by an even number of transpositions, and negative if it requires an odd number of transpositions. The statement above would then be that the determinant is unchanged if the basis vectors are related by a positive permutation, and picks a sign if they are related by a negative permutation.

4.2 Multiplying k-forms: the wedge product

With the introduction of k-forms we can now introduce a new operation: we can multiply differential forms. This operation is called the "wedge product", which is what we study in this section.

Objectives

You should be able to:

- Determine the wedge product of two k-forms in $\mathbb{R}^3$.
- Relate the wedge product of two one-forms to the cross-product of the associated vector fields, and the wedge product of a one-form and a two-form to the dot product of the associated vector fields.
- State and use general properties of the wedge product for k-forms.

4.2.1 Multiplying k-forms: the wedge product

When we introduced one-forms in Section 2.1, we explained how we can add two one-forms to get another one-form, and how we can multiply a one-form by a function to get another one-form. Let us first state that the addition property obviously holds for two- and three-forms as well: the sum of two two-forms is a two-form, and the sum of two three-forms is a three-form. Note that it doesn't really make sense to add a one-form with a two-form, etc.

When we introduced one-forms we did not however talk about multiplying two one-forms together. To do this, we need the full theory of k-forms, as multiplying two one-forms will give us a two-form. The operation of multiplying k-forms is called the "wedge product", and denoted by $\wedge$, to which we now turn to.

Definition 4.2.1 The wedge product. Consider a simple k-form ω and a simple m-form η on an open subset $U \subseteq \mathbb{R}^n$ of the form:

$$\omega = f\ dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \qquad \eta = g\ dx_{j_1} \wedge \ldots \wedge dx_{j_m},$$

for smooth functions $f, g : U \to \mathbb{R}$. Then the **wedge product** $\omega \wedge \eta$ is a $(k+m)$-form defined by:

$$\omega \wedge \eta = fg\ dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_m}.$$

Note that the order is important here on the right, since we know that exchanging two dx_i's will pick a sign, by the definition of basic forms.

The wedge product of arbitrary k- and m-forms is then defined by linearity, i.e. by distributing the wedge product term by term in the expression of the forms as sums of terms.

$\Diamond$

This definition looks complicated, but in the end it is fairly simple. It is easier to understand it by working through some examples.

Example 4.2.2 The wedge product of two one-forms. Consider the wedge product of the two one-forms $\omega = x\ dx + x\ dy + e^z\ dz$ and $\eta = y\ dx + x\ dy + z\ dz$ on $\mathbb{R}^3$. By definition, the result is the two-form given by:

$$\begin{aligned}\omega \wedge \eta =& (x\ dx + x\ dy + e^z\ dz) \wedge (y\ dx + x\ dy + z\ dz)\\ =& xy\ dx \wedge dx + x^2\ dx \wedge dy + xz\ dx \wedge dz + xy\ dy \wedge dx + x^2 dy \wedge dy + xz\ dy \wedge dz\\ &+ ye^z\ dz \wedge dx + xe^z\ dz \wedge dy + ze^z\ dz \wedge dz\\ =& (xz - xe^z) dy \wedge dz + (-xz + ye^z) dz \wedge dx + (x^2 - xy)\ dx \wedge dy,\end{aligned}$$

where in the last line we used the fact that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$ and $dx \wedge dy = -dy \wedge dx$, $dx \wedge dz = -dz \wedge dx$, and $dy \wedge dz = -dz \wedge dy$. □

Example 4.2.3 The wedge product of a one-form and a two-form. Consider the wedge product of the one-form $\omega = xy\ dx + y\ dy$ and the two-form $\eta = z\ dy \wedge dz$ on $\mathbb{R}^3$. The result is the three-form:

$$\begin{aligned}\omega \wedge \eta =& (xy\ dx + y\ dy) \wedge (z\ dy \wedge dz)\\ =& xyz\ dx \wedge dy \wedge dz + yz\ dy \wedge dy \wedge dz\\ =& xyz\ dx \wedge dy \wedge dz,\end{aligned}$$

where we used the fact that $dy \wedge dy \wedge dz = 0$ since it has repeated factors. □

Example 4.2.4 The wedge product of a zero-form and a k-form. The wedge product with a zero-form is just a standard product, since a zero-form is just a function. For instance, given the zero-form $f = x$ and the two-form $\omega = y dy \wedge dz$ on $\mathbb{R}^3$, the wedge product is the two-form:

$$f \wedge \omega = xy\ dy \wedge dz.$$

We usually write $f\omega$, without the $\wedge$ symbol, when we multiply with a zero-form, since it is just a function.

Note that this generalizes the statement that a one-form mutiplied by a function is another one-form; this is also true for a two-form and a three-form. □

Remark 4.2.5 These examples pretty much exhaust the possible non-zero wedge products in $\mathbb{R}^3$. Indeed, since we know that k-forms with $k \geq 4$ necessarily vanish in $\mathbb{R}^3$, this means that the only possible non-zero wedge products are:

- A zero-form (i.e. a function) with a k-form, with $k \in \{0, 1, 2, 3\}$;
- A one-form with a one-form, which gives a two-form;
- A one-form with a two-form, which gives a three-form.

All other wedge products will necessarily vanish.

Now that we are familiar with the wedge product, a natural question is whether it is "commutative". Pick two differential forms ω and η. Is $\omega \wedge \eta$ equal to $\eta \wedge \omega$? The answer is no, not quite! The precise statement is the following lemma.

Lemma 4.2.6 Comparing $\omega \wedge \eta$ to $\eta \wedge \omega$. *Let ω be a k-form and η be an m-form. Then*

$$\omega \wedge \eta = (-1)^{km} \eta \wedge \omega.$$

In other words:

- *If either k or m is even, then $\omega \wedge \eta = \eta \wedge \omega$, and the wedge product is commutative.*
- *If both k and m are odd, then $\omega \wedge \eta = -\eta \wedge \omega$, and the wedge product is anti-commutative.*

In particular, if ω is a k-form with k odd, then $\omega \wedge \omega = 0$.

Proof. This statement follows from the fact that exchanging two dx_i's in a basic form picks a sign. More precisely, let us first assume that ω and η take the simple forms:

$$\omega = f\ dx_{i_1} \wedge \ldots dx_{i_k}, \qquad \eta = g\ dx_{j_1} \wedge \ldots \wedge dx_{j_m}.$$

Then

$$\omega \wedge \eta = fg\ \ (dx_{i_1} \wedge \ldots dx_{i_k}) \wedge (dx_{j_1} \wedge \ldots \wedge dx_{j_m}),$$

while

$$\eta \wedge \omega = fg\ \ (dx_{j_1} \wedge \ldots \wedge dx_{j_m}) \wedge (dx_{i_1} \wedge \ldots dx_{i_k}).$$

To relate the second expression to the first, we need to move the dx_j's to the left of the dx_i's. We first move dx_{i_1} to the left. Each time we pass a dx_j, we pick a sign. So in the end we get:

$$\eta \wedge \omega = fg(-1)^m\ dx_{i_1} \wedge (dx_{j_1} \wedge \ldots \wedge dx_{j_m}) \wedge (dx_{i_2} \wedge \ldots dx_{i_k}).$$

We do the same thing with dx_{i_2}, moving it pass all the dx_j's, and so on all the way to dx_{i_k}. The final result is

$$\begin{aligned}\eta \wedge \omega =& fg(-1)^{km}\ dx_{i_1} \wedge \ldots dx_{i_k} \wedge (dx_{j_1} \wedge \ldots \wedge dx_{j_m})\\ =&(-1)^{km}\omega \wedge \eta.\end{aligned}$$

Finally, the statement is proved for general differential forms by doing this manipulation term by term after distributing the wedge product. ■

4.2.2 The wedge product and vector calculus

In Table 4.1.11 we established a dictionary between differential forms in $\mathbb{R}^3$ and vector calculus concepts. We can extend this dictionary to understand the concept of wedge product in vector calculus.

As we saw, there are really only three types of non-zero wedge products in $\mathbb{R}^3$:

1. The wedge product of a zero-form with a k-form with $k \in \{0, 1, 2, 3\}$;
2. A one-form with a one-form;
3. A one-form with a two-form.

Let us now translate each of those operations in the language of vector calculus.

First, multiplying a zero-form by a k-form is just multiplying the k-form by a function. So the same is true in vector calculus: we multiply the corresponding vector field or function by a function. No big deal.

Multiplying two one-forms is interesting though. Let $\omega = f_1 dx + f_2\ dy + f_3\ dz$ and $\eta = g_1\ dx + g_2\ dy + g_3\ dz$. Then the wedge product is

$$\omega \wedge \eta = (f_2 g_3 - f_3 g_2) dy \wedge dz + (f_3 g_1 - f_1 g_3) dz \wedge dx + (f_1 g_2 - f_2 g_1) dx \wedge dy.$$

In terms of the associated vector fields

$$\mathbf{F} = (f_1, f_2, f_3), \qquad \mathbf{G} = (g_1, g_2, g_3),$$

what we are doing is constructing a new vector field, let's call it $\mathbf{H}$ for the time being, associated to $\omega \wedge \eta$, with component functions given by (according to the dictionary established in Table 4.1.11 for relating a two-form to a vector field):

$$\mathbf{H} = (f_2 g_3 - f_3 g_2, f_3 g_1 - f_1 g_3, f_1 g_2 - f_2 g_1) .$$

What is this vector field? It is nothing but the cross-product of the vector fields $\mathbf{F}$ and $\mathbf{G}$! Indeed,

$$\mathbf{F} \times \mathbf{G} = (f_2 g_3 - f_3 g_2, f_3 g_1 - f_1 g_3, f_1 g_2 - f_2 g_1) .$$

Therefore, we end up with the following statement:

Lemma 4.2.7 The wedge product of two one-forms is the cross-product of the associated vector fields. *If ω and η are two one-forms on $U \subseteq \mathbb{R}^3$, with associated vector fields $\mathbf{F}$ and $\mathbf{G}$, then the vector field associated to the two-form $\omega \wedge \eta$ via the dictionary in Table 4.1.11 is the cross-product $\mathbf{F} \times \mathbf{G}$.*

Finally, we consider the wedge product of a one-form and a two-form. Let $\omega = f_1\, dx + f_2\, dy + f_3\, dz$ and $\eta = g_1\, dy \wedge dz + g_2\, dz \wedge dx + g_3\, dx \wedge dy$. Then the wedge product is the three-form:

$$\omega \wedge \eta = (f_1 g_1 + f_2 g_2 + f_3 g_3)\, dx \wedge dy \wedge dz.$$

According to the dictionary in Table 4.1.11, we thus conclude that the function associated to the three-form $\omega \wedge \eta$ is nothing but the dot product of the two vector fields $\mathbf{F}$ and $\mathbf{G}$ associated to ω and η respectively:

$$\mathbf{F} \cdot \mathbf{G} = f_1 g_1 + f_2 g_2 + f_3 g_3.$$

Neat! So we get the following result:

Lemma 4.2.8 The wedge product of a one-form and a two-form is the dot product of the associated vector fields. *If ω is a one-form and η a two-form on $U \subseteq \mathbb{R}^3$, with associated vector fields $\mathbf{F}$ and $\mathbf{G}$ via the dictionary in Table 4.1.11, then the function associated to the three-form $\omega \wedge \eta$ is the dot product $\mathbf{F} \cdot \mathbf{G}$.*

These two lemmas justify the ordering in the definition of two-form mentioned in Remark 4.1.9.

We can summarize the dictionary between the wedge product and vector products in the following two tables:

Table 4.2.9 Dictionary between the wedge product of two one-forms in $\mathbb{R}^3$ and vector calculus concepts

Differential form concept	Vector calculus concept
1-form ω	vector field $\mathbf{F}$
1-form η	vector field $\mathbf{G}$
2-form $\omega \wedge \eta$	vector field $\mathbf{F} \times \mathbf{G}$

Table 4.2.10 Dictionary between the wedge product of a one-form and a two-form in $\mathbb{R}^3$ and vector calculus concepts

Differential form concept	Vector calculus concept
1-form ω	vector field $\mathbf{F}$
2-form η	vector field $\mathbf{G}$
3-form $\omega \wedge \eta$	vector field $\mathbf{F} \cdot \mathbf{G}$

4.2.3 Exercises

1. Let $\omega = e^x\ dx + y\ dz$ and $\eta = xy\ dx + z\ dy + y\ dz$. Find $\omega \wedge \eta$, and write your result in standard form.

Solution. We find (using the fact that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$):

$$\begin{aligned}\omega \wedge \eta =&(e^x\ dx + y\ dz) \wedge (xy\ dx + z\ dy + y\ dz)\\ =&ze^x\ dx \wedge dy + ye^x\ dx \wedge dz + xy^2\ dz \wedge dx + yz\ dz \wedge dy\\ =&-yz\ dy \wedge dz + (xy^2 - ye^x)\ dz \wedge dx + ze^x\ dx \wedge dy.\end{aligned}$$

2. Let $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$ and $\eta = x\ dx + y\ dy + z\ dz$. Find $\omega \wedge \eta$, and write your result in standard form.

Solution. We find (using the fact that any wedge product with two repeated dx, dy or dz is zero):

$$\begin{aligned}\omega \wedge \eta =&(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy) \wedge (x\ dx + y\ dy + z\ dz)\\ =&x^2\ dy \wedge dz \wedge dx + y^2\ dz \wedge dx \wedge dy + z^2\ dx \wedge dy \wedge dz\\ =&(x^2 + y^2 + z^2)\ dx \wedge dy \wedge dz,\end{aligned}$$

where we used the fact that $dy \wedge dz \wedge dx = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$.

3. Let $\omega = f_1\ dx + f_2\ dy$ and $\eta = g_1\ dx + g_2\ dy$ be one-forms on $\mathbb{R}^2$, with associated vector fields $\mathbf{F} = (f_1, f_2)$ and $\mathbf{G} = (g_1, g_2)$. Show that

$$\omega \wedge \eta = \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} dx \wedge dy.$$

Solution. Let $\omega = f_1\ dx + f_2\ dy$, and $\eta = g_1\ dx + g_2\ dy$. The associated vector fields are $\mathbf{F} = (f_1, f_2)$, $\mathbf{G} = (g_1, g_2)$. We calculate the wedge product:

$$\begin{aligned}\omega \wedge \eta =&(f_1\ dx + f_2\ dy) \wedge (g_1\ dx + g_2\ dy)\\ =&f_1 g_2\ dx \wedge dy + f_2 g_1\ dy \wedge dx\\ =&(f_1 g_2 - f_2 g_1) dx \wedge dy\\ =&\det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} dx \wedge dy.\end{aligned}$$

We note that this determinant could be taken as a "definition" of what "cross-product" of vector fields means in $\mathbb{R}^2$. Note that the result however is a function, not a vector field. From the point of view of differential forms, we could define the "cross-product" of vector fields in $\mathbb{R}^n$ as follows: take two one-forms ω and η on $\mathbb{R}^n$, with associated vector fields $\mathbf{F}$ and $\mathbf{G}$. We would define the "cross-product" of the vector fields as being given by the two-form $\omega \wedge \eta$ in $\mathbb{R}^n$. Note that only on $\mathbb{R}^3$ can we associate to the result a new vector field (this is because of Hodge duality between one-forms and two-forms in $\mathbb{R}^3$, see Section 4.8). For instance, in $\mathbb{R}^4$ a two-form has 6 component functions (there are 6 independent non-vanishing basic two-forms in $\mathbb{R}^4$), so it cannot be associated to a vector field.

4. Let $\omega = f\ dx + g\ dy + h\ dz$ be an arbitrary one-form on $\mathbb{R}^3$. By doing an explicit calculation, show that $\omega \wedge \omega = 0$.

Solution. We find:

$$\begin{aligned}\omega \wedge \omega =&(f\ dx + g\ dy + h\ dz) \wedge (f\ dx + g\ dy + h\ dz)\\ =&fg\ dx \wedge dy + fh\ dx \wedge dz + gf\ dy \wedge dx + gh\ dy \wedge dz + hf\ dz \wedge dx + hg\ dz \wedge dy\\ =&(gh - hg)\ dy \wedge dz + (hf - fh)\ dz \wedge dx + (fg - gf)\ dx \wedge dy.\end{aligned}$$

Since functions commute with each other, i.e. $gh = hg$, $hf = fh$, $fg = gf$, we conclude that $\omega \wedge \omega = 0$.

5. Find the cross-product of the vectors $\mathbf{F} = (1, 1, 0)$ and $\mathbf{G} = (0, 2, 3)$ in $\mathbb{R}^3$ by computing the wedge product of the associated one-forms.

Solution. The one-forms associated to $\mathbf{F}$ and $\mathbf{G}$ are $\omega = dx + dy$ and $\eta = 2dy + 3dz$. We calculate the wedge product:

$$\begin{aligned}\omega \wedge \eta =&(dx + dy) \wedge (2dy + 3dz)\\ =&2dx \wedge dy + 3dx \wedge dz + 3dy \wedge dz\\ =&3dy \wedge dz - 3dz \wedge dx + 2dx \wedge dy\end{aligned}$$

According to the dictionary Table 4.1.11, the vector field associated to this two-form is

$$\mathbf{H} = \mathbf{F} \times \mathbf{G} = (3, -3, 2).$$

This is indeed the cross-product, as you can calculate using standard formulae from linear algebra. For instance, you may have seen the formula:

$$\begin{aligned}\mathbf{F} \times \mathbf{G} =& \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{pmatrix}\\ =& \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}\\ =&3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k},\end{aligned}$$

which, in component notation, reads $(3, -3, 2)$.

6. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be vectors in $\mathbb{R}^3$. In your linear algebra course you may have seen that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$

Prove this property by looking at the wedge product of the three one-forms ω, η, λ associated to the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

Solution. As discussed in Lemma 4.2.7 and Lemma 4.2.8, we know that

$$\omega \wedge \eta \wedge \lambda = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) dx \wedge dy \wedge dz,$$

$$\eta \wedge \lambda \wedge \omega = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) dx \wedge dy \wedge dz,$$

and

$$\lambda \wedge \omega \wedge \eta = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) dx \wedge dy \wedge dz.$$

But since λ, ω, η are one-forms, we know that $\omega \wedge \eta = -\eta \wedge \omega$, $\eta \wedge \lambda = -\lambda \wedge \eta$, and $\omega \wedge \lambda = -\lambda \wedge \omega$. Therefore,

$$\omega \wedge \eta \wedge \lambda = \eta \wedge \lambda \wedge \omega = \lambda \wedge \omega \wedge \eta,$$

and the statement is proved. In other words, it follows directly from anti-commutativity of the wedge product of one-forms.

4.3 Differentiating k-forms: the exterior derivative

We know how to add k-forms, and now also how to mutiply k-forms, thanks to the notion of wedge product. In this section we study how we can "differentiate" k-forms, using the notion of exterior derivative, which generalizes the differential of a function introduced in Definition 2.2.1.

Objectives

You should be able to:

- Define the exterior derivative of a k-form, focusing on zero-, one- and two-forms in $\mathbb{R}^3$.
- State and use the graded product rule for the exterior derivative.
- Show that applying the exterior derivative twice always gives zero.

4.3.1 The exterior derivative

Let us start by recalling the definition of the differential of a function $f : U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^3$, from Definition 2.2.1. The differential df is the one-form on U given by

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

Since we now think of f as a zero-form, we see that the operator d takes a zero-form and outputs a one-form. Our goal is to generalize this operation, which we will now call the "exterior derivative", to take a k-form and ouput a $(k+1)$-form.

Definition 4.3.1 The exterior derivative of a k-form. The **exterior derivative of a**

zero-form f on $U \subset \mathbb{R}^n$ is the one-form df on U given by:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i,$$

which is the same thing as the differential introduced in Definition 2.2.1.

The **exterior derivative of a k-form** $\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ on $U \subset \mathbb{R}^n$ is the $(k+1)$-form $d\omega$ on U given by:

$$d\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} d(f_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $d(f_{i_1 \cdots i_k})$ means the exterior derivative of the zero-form (function) $f_{i_1 \cdots i_k}$. In other words, we are applying the exterior derivative d to the component functions of ω. $\Diamond$

This definition may seem a little daunting because of the summations, so let us be more explicit for k-forms in $\mathbb{R}^3$.

Lemma 4.3.2 The exterior derivative in $\mathbb{R}^3$.

1. *If f is a zero-form on $U \subseteq \mathbb{R}^3$, then its exterior derivative df is the one-form:*

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

2. *If $\omega = f\ dx + g\ dy + h\ dz$ is a one-form on $U \subseteq \mathbb{R}^3$, then its exterior derivative $d\omega$ is the two-form:*

$$\begin{aligned} d\omega =& d(f) \wedge dx + d(g) \wedge dy + d(h) \wedge dz \\ =& \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy. \end{aligned}$$

3. *If $\eta = f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$ is a two-form on $U \subseteq \mathbb{R}^3$, then its exterior derivative $d\eta$ is the three-form:*

$$\begin{aligned} d\eta =& d(f) \wedge dy \wedge dz + d(g) \wedge\ dz \wedge dx + d(h) \wedge dx \wedge dy. \\ =& \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz. \end{aligned}$$

And that's it. In particular, the exterior derivative of a three-form on $\mathbb{R}^3$ must vanish, since it would give a four-form, but all k-forms with $k \geq 4$ necessarily vanish on $\mathbb{R}^3$ by Lemma 4.1.6.

Proof. We start with Definition 4.3.1 restricted to the case with $n = 3$. For the exterior derivative of a zero-form, the statement is obvious. For the exterior derivatives of a one-form and a two-form, all we have to do is evaluate the exterior derivatives $d(f), d(g)$ and $d(h)$ of the zero-form f, g, h, and rearrange terms using Lemma 4.1.6. ∎

Note that you certainly should not aim at learning these formulae by heart. The whole point is precisely that you don't need to learn these formulae! All you need to remember is that, to evaluate the exterior derivative of a k-form, you act with the exterior derivative on the component functions of the k-form. This may be clearer with examples.

Example 4.3.3 The exterior derivative of a zero-form on $\mathbb{R}^3$. We are already familiar with the calculation of the exterior derivative of a zero-form, since this is the same thing as the calculation of the differential of a function that we defined in Definition 2.2.1. But let us give an example here for completeness.

Let $f(x, y, z) = y \ln(x) + z$ be a smooth function on $U = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}$. Its exterior derivative is the one-form df on U given by:

$$\begin{aligned} df =& \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz \\ =& \frac{y}{x}\ dx + \ln(x)\ dy + dz. \end{aligned}$$

□

Example 4.3.4 The exterior derivative of a one-form on $\mathbb{R}^3$. Let $\omega = xy\ dx + (z + y)\ dy + xyz\ dz$ be a one-form on $\mathbb{R}^3$. Its exterior derivative is the two-form $d\omega$ on $\mathbb{R}^3$ given by:

$$\begin{aligned} d\omega =& d(xy) \wedge dx + d(z + y) \wedge dy + d(xyz) \wedge dz \\ =& (y\ dx + x\ dy) \wedge dx + (dy + dz) \wedge dy + (yz\ dx + xz\ dy + xy\ dz) \wedge dz \\ =& x\ dy \wedge dx + dz \wedge dy + yz\ dx \wedge dz + xz\ dy \wedge dz \\ =& (xz - 1)\ dy \wedge dz - yz\ dz \wedge dx - x\ dx \wedge dy. \end{aligned}$$

□

Example 4.3.5 The exterior derivative of a two-form on $\mathbb{R}^3$. Let $\omega = (x^2 + y^2)\ dy \wedge dz + \sin(z)\ dz \wedge dx + \cos(xy)\ dx \wedge dy$ be a two-form on $\mathbb{R}^3$. Its exterior derivative is the three-form $d\omega$ on $\mathbb{R}^3$ given by:

$$\begin{aligned} d\omega =& d(x^2 + y^2) \wedge dy \wedge dz + d(\sin(z)) \wedge dz \wedge dx + d(\cos(xy)) \wedge dx \wedge dy \\ =& (2x\ dx + 2y\ dy) \wedge dy \wedge dz + (\cos(z)\ dz) \wedge dz \wedge dx + (-y \sin(xy)\ dx - x \sin(xy)\ dy) \wedge dx \wedge dy \\ =& 2x\ dx \wedge dy \wedge dz. \end{aligned}$$

We see that going from the second line to the third line, most terms vanish, since anytime we take the wedge of dx with itself we get zero, and same for dy and dz. □

To end this section, we note that the exterior derivative is linear. If ω and η are two k-forms, and $a, b \in \mathbb{R}$, then

$$d(a\omega + b\eta) = ad\omega + bd\eta.$$

This is proven in Exercise 4.3.4.3.

4.3.2 The graded product rule

One of the most fundamental properties of the derivative is the product rule:

$$\frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx}.$$

Now that we have defined the exterior derivative for differential forms, and that we know how to multiply differential forms using the wedge product, we could ask whether the exterior derivative satisfies a similar product rule with respect to the wedge product. It turns out that it does, but with a twist (or more precisely a sign). We call this the "graded product rule" for the exterior derivative.

Lemma 4.3.6 The graded product rule for the exterior derivative. *Let ω be a k-form and η be an l-form on $U \subseteq \mathbb{R}^n$. Then:*

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta).$$

Proof. First, we show that it is true for zero-forms. If ω and η are zero-forms (that is $k = l = 0$), that is functions $\omega = f$ and $\eta = g$, then

$$\begin{aligned} d(\omega \wedge \eta) =& d(fg) \\ =& \sum_{i=1}^n \frac{\partial}{\partial x_i}(fg)\ dx_i \\ =& \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right)\ dx_i \\ =& d(f)g + f d(g). \end{aligned}$$

To prove the general statement we unfortunately need to use lots of summations. Let us introduce the following notation for the k-form ω and the l-form η:

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} w_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad \eta = \sum_{1 \leq j_1 < \cdots < j_l \leq n} h_{j_1 \cdots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

The exterior derivative of the wedge product is:

$$\begin{aligned} d(\omega \wedge \eta) =& \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{1 \leq j_1 < \cdots < j_l \leq n} d(w_{i_1 \cdots i_k} h_{j_1 \cdots j_l}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ =& \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{1 \leq j_1 < \cdots < j_l \leq n} (d(w_{i_1 \cdots i_k}) h_{j_1 \cdots j_l} + w_{i_1 \cdots i_k} d(h_{j_1 \cdots j_l})) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}. \end{aligned}$$

To go from the first to the second line, we used the fact that the product rule is satisfied for the exterior derivative of the product of zero-forms, as shown above.

Now let us study the terms on the right-hand-side of the equation we just obtained. First, we get:

$$\begin{aligned} &\sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{1 \leq j_1 < \cdots < j_l \leq n} d(w_{i_1 \cdots i_k}) h_{j_1 \cdots j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &\qquad = \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} d(w_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \wedge \left(\sum_{1 \leq j_1 < \cdots < j_l \leq n} h_{j_1 \cdots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right) \\ &\qquad = d(\omega) \wedge \eta \end{aligned}$$

All that we did to go from the first line to the second line is move the zero-form (or function) $h_{j_1 \cdots j_l}$ to the right of the dx_i's, which we can do since it is a zero-form and hence commutes with the dx_i's by Lemma 4.2.6.

That takes care of the first set of terms. The remaining ones take the form

$$\sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{1\le j_1<\cdots<j_l\le n} w_{i_1\cdots i_k} d(h_{j_1\cdots j_l})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}.$$

We would like to do the same and commute $d(h_{j_1\cdots j_l})$ to the right of the dx_i's, so that we can identify this term as $\omega\wedge d(\eta)$. However, $d(h_{j_1\cdots j_l})$ is a one-form, and thus by Lemma 4.2.6 $d(h_{j_1\cdots j_l})\wedge dx_i = -dx_i\wedge d(h_{j_1\cdots j_l})$. So every time we commute $d(h_{j_1\cdots j_l})$ past a dx_i, we pick a sign. Since there are k dx_i's in this expression, this tells us that it is equal to

$$(-1)^k\omega\wedge d(\eta).$$

Putting all this together, we end up with the statement that

$$d(\omega\wedge\eta) = d(\omega)\wedge\eta + (-1)^k\omega\wedge d(\eta),$$

which is the graded product rule stated in the lemma. ■

It is very important not to forget the sign in the graded product rule!

Example 4.3.7 The exterior derivative of the wedge product of two one-forms. Let $\omega = xy\ dz$ and $\eta = (y+z)\ dx + 2dy$ be two one-forms on $\mathbb{R}^3$. Suppose that we want to calculate the exterior derivative of the wedge product $\omega\wedge\eta$. There are two ways we can do that: we can first find an explicit expression for the wedge product, and then take the exterior derivative, or we can use the graded product rule. In this example we show that both calculations give the same result, as they should.

Let us first calculate the wedge product explicitly and take its exterior derivative. We have:

$$\begin{aligned}\omega\wedge\eta =&(xy\ dz)\wedge((y+z)\ dx + 2\ dy)\\ =&-2xydy\wedge dz + xy(y+z)dz\wedge dx.\end{aligned}$$

We then calculate the exterior derivative, which gives the following three-form:

$$\begin{aligned}d(\omega\wedge\eta) =&d(-2xy)\wedge dy\wedge dz + d(xy(y+z))\wedge dz\wedge dx\\ =&(-2y\ dx - 2x\ dy)\wedge dy\wedge dz + (y(y+z)\ dx + (2xy+xz)\ dy + xy\ dz)\wedge dz\wedge dx\\ =&(-2y + (2xy+xz))dx\wedge dy\wedge dz,\end{aligned}$$

where we used the fact that $dy\wedge dz\wedge dx = dx\wedge dy\wedge dz$.

Let us now calculate the same exterior derivative but using the graded product rule. Since ω is a one-form, we have:

$$d(\omega\wedge\eta) = d(\omega)\wedge\eta - \omega\wedge d(\eta).$$

We calculate:

$$\begin{aligned}d(\omega) =&d(xy)\wedge dz\\ =&ydx\wedge dz + xdy\wedge dz,\end{aligned}$$

and

$$d(\eta) =d(y+z)\wedge dx + d(2)\wedge dy$$

$$=dy \wedge dx + dz \wedge dx,$$

since $d(2) = 0$. Putting this together, we get:

$$\begin{aligned} d(\omega \wedge \eta) =& (ydx \wedge dz + xdy \wedge dz) \wedge ((y+z)\ dx + 2\ dy) - (xy\ dz) \wedge (dy \wedge dx + dz \wedge dx) \\ =& (-2y + x(y+z) + xy) dx \wedge dy \wedge dz, \end{aligned}$$

which is indeed the same result as obtained above. □

Remark 4.3.8 In $\mathbb{R}^3$, the graded product rule can be split into the four following non-vanishing cases.

1. If $\omega = f$ is a zero-form (in which case we write $f \wedge \eta = f\eta$ as usual when multiplying with a function) and $\eta = g$ is a zero-form, then
$$d(fg) = d(f)g + fd(g).$$
2. If $\omega = f$ is a zero-form and η is a one-form, then
$$d(f\eta) = d(f) \wedge \eta + fd(\eta).$$
3. If $\omega = f$ is a zero-form and η is a two-form, then
$$d(f\eta) = d(f) \wedge \eta + fd(\eta).$$
4. If ω is a one-form and η is a one-form, then
$$d(\omega \wedge \eta) = d(\omega) \wedge \eta - \omega \wedge d(\eta).$$

4.3.3 $d^2 = 0$

There's another fundamental property of the exterior derivative: if we apply the exterior derivative twice, we always get zero. This may seem surprising, as this is certainly not true for the ordinary derivative d/dx, but it is true for the exterior derivative because of antisymmetry of the wedge product. More precisely:

Lemma 4.3.9 $d^2 = 0$**.** *Let ω be a k-form on $U \subseteq \mathbb{R}^n$. Then*

$$d(d(\omega)) = 0.$$

In other words, if we apply the exterior derivative twice on any differential form, we always get zero. We often abbreviate this statement as $d^2 = 0$, meaning that applying the exterior derivative twice always gives zero.

Proof. We write $\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} w_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. We have:

$$d(d(\omega)) = d\left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} d(w_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right)$$

$$\begin{aligned}
&= d\left(\sum_{1\le i_1<\cdots<i_k\le n}\sum_{\alpha=1}^{n}\frac{\partial w_{i_1\cdots i_k}}{\partial x_\alpha}dx_\alpha\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)\\
&= \sum_{1\le i_1<\cdots<i_k\le n}\sum_{\alpha=1}^{n} d\left(\frac{\partial w_{i_1\cdots i_k}}{\partial x_\alpha}\right)\wedge dx_\alpha\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\
&= \sum_{1\le i_1<\cdots<i_k\le n}\left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}dx_\beta\wedge dx_\alpha\right)\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}.
\end{aligned}$$

If we can show that the term in brackets in the last line is zero, then clearly $d(d(\omega))=0$. We have:

$$\begin{aligned}
\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}dx_\beta\wedge dx_\alpha &= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\alpha\partial x_\beta}dx_\beta\wedge dx_\alpha\\
&= -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\alpha\partial x_\beta}dx_\alpha\wedge dx_\beta.
\end{aligned}$$

In the first, line, we used the fact that $\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}=\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\alpha\partial x_\beta}$ by the Clairaut-Schwarz theorem, since the coefficient functions $w_{i_1\cdots i_k}$ are assumed to be smooth. In the second line, we used the fact that $dx_\beta\wedge dx_\alpha=-dx_\alpha\wedge dx_\beta$, by Lemma 4.1.6. Finally, in the summation on the right-hand-side, we can simply rename α to be β, and β to be α, since those are indices that are summed over, and hence we can give them the name we want. We end up with the statement that

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}dx_\beta\wedge dx_\alpha = -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}dx_\beta\wedge dx_\alpha.$$

But the only two-form that is equal to minus itself is the zero two-form, that is

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial^2 w_{i_1\cdots i_k}}{\partial x_\beta\partial x_\alpha}dx_\beta\wedge dx_\alpha = 0.$$

Therefore we conclude that $d(d(\omega))=0$. ■

4.3.4 Exercises

1. Let $\omega = xy\ dx + \frac{y}{x}\ dy + \frac{z}{x}\ dz$ on $U=\{(x,y,z)\in\mathbb{R}^3 \mid x\neq 0\}$. Find the two-form $d\omega$ and write your result in standard form. What is the vector field associated to $d\omega$?

Solution. From the definition, we find:

$$\begin{aligned}
d\omega &= d(xy)\wedge dx + d\left(\frac{y}{x}\right)\wedge dy + d\left(\frac{z}{x}\right)\wedge dz\\
&= y\ dx\wedge dx + x\ dy\wedge dx + \frac{1}{x}dy\wedge dy - \frac{y}{x^2}dx\wedge dy + \frac{1}{x}\ dz\wedge dz - \frac{z}{x^2}\ dx\wedge dz\\
&= \frac{z}{x^2}\ dz\wedge dx - \left(x+\frac{y}{x^2}\right)dx\wedge dy.
\end{aligned}$$

Using the dictionary Table 4.1.11, the vector field associated to the two-form $d\omega$ is

$$\mathbf{F}(x,y,z) = \left(0, \frac{z}{x^2}, -x - \frac{y}{x^2}\right).$$

2. Find the three-form $d\omega$ if $\omega = xyz(dy \wedge dz + dz \wedge dx + dx \wedge dy)$, and write your result in standard form.

Solution. We find:

$$\begin{aligned} d\omega =& d(xyz) \wedge (dy \wedge dz + dz \wedge dx + dx \wedge dy) \\ =& (yz\ dx + xz\ dy + xy\ dz) \wedge (dy \wedge dz + dz \wedge dx + dx \wedge dy) \\ =& yz\ dx \wedge dy \wedge dz + xz\ dy \wedge dz \wedge dx + xy\ dz \wedge dx \wedge dy \\ =& (yz + xz + xy)\, dx \wedge dy \wedge dz. \end{aligned}$$

3. Show that the exterior derivative is linear. That is, if ω and η are two k-forms, and $a, b \in \mathbb{R}$, then

$$d(a\omega + b\eta) = ad\omega + bd\eta.$$

Solution. First, we show that the property holds if ω and η are 0-forms. Since those are simply functions, let us write them as f and g. Then:

$$\begin{aligned} d(af + bg) =& \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(af + bg)\ dx_i \\ =& \sum_{i=1}^{n} \left(a\frac{\partial f}{\partial x_i} + b\frac{\partial g}{\partial x_i}\right)\ dx_i \\ =& a\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\ dx_i + b\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\ dx_i \\ =& adf + bdg. \end{aligned}$$

For the second equality we used the fact that partial derivatives are linear.

Now we can prove the general case. Suppose that ω and η are k-forms on $U \subseteq \mathbb{R}^n$. Let

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and

$$\eta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Then

$$\begin{aligned} d(a\omega + b\eta) =& \sum_{1 \leq i_1 < \cdots < i_k \leq n} d(af_{i_1 \cdots i_k} + bg_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ =& \sum_{1 \leq i_1 < \cdots < i_k \leq n} (a\ df_{i_1 \cdots i_k} + b\ dg_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ =& a \sum_{1 \leq i_1 < \cdots < i_k \leq n} df_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \end{aligned}$$

$$
\begin{aligned}
&+ b \sum_{1\le i_1<\cdots<i_k\le n} dg_{i_1\cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\\
=&a d\omega + b d\eta.
\end{aligned}
$$

In the second equality we used the calculation above that showed that linearity holds for 0-forms.

4. Let $\omega = xe^y\ dx + z\ dy + ye^x\ dz$. Show by explicit calculation that $d^2\omega = 0$.

Solution. Of course, we know that $d^2\omega = 0$ since this is true for any differential form ω, as proven in Lemma 4.3.9. But let us show that it is true by explicit calculation for this particular one-form ω.

We first calculate the two-form $d\omega$:

$$
\begin{aligned}
d\omega =&d(xe^y) \wedge dx + dz \wedge dy + d(ye^x) \wedge dz\\
=&xe^y\ dy \wedge dx + e^y\ dx \wedge dx + dz \wedge dy + ye^x\ dx \wedge dz + e^x\ dy \wedge dz\\
=&(e^x - 1)\ dy \wedge dz - ye^x\ dz \wedge dx - xe^y\ dx \wedge dy.
\end{aligned}
$$

We then calculate the three-form $d^2\omega$:

$$
\begin{aligned}
d^2\omega =&d(d\omega)\\
=&d(e^x - 1) \wedge dy \wedge dz - d(ye^x) \wedge dz \wedge dx - d(xe^y) \wedge dx \wedge dy\\
=&e^x\ dx \wedge dy \wedge dz - ye^x\ dx \wedge dz \wedge dx - e^x\ dy \wedge dz \wedge dx - xe^y\ dy \wedge dx \wedge dy\\
&- e^y\ dx \wedge dx \wedge dy\\
=&(e^x - e^x) dx \wedge dy \wedge dz\\
=&0,
\end{aligned}
$$

as expected.

5. Let $\omega = (x^2 - y^2)\ dx + y\ dz$ and $\eta = (x^2 + y^2)\ dy + y\ dz$. By explicit calculation, show that

$$
d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta,
$$

which is consistent with the graded product rule Lemma 4.3.6 since ω is a one-form.

Solution. We need to calculate $d(\omega \wedge \eta)$, $d\omega \wedge \eta$, and $\omega \wedge d\eta$. First, we calculate $\omega \wedge \eta$:

$$
\begin{aligned}
\omega \wedge \eta =&((x^2 - y^2)\ dx + y\ dz) \wedge ((x^2 + y^2)\ dy + y\ dz)\\
=&(x^4 - y^4) dx \wedge dy + y(x^2 - y^2)\ dx \wedge dz + y(x^2 + y^2)\ dz \wedge dy\\
=&-y(x^2 + y^2)\ dy \wedge dz - y(x^2 - y^2)\ dz \wedge dx + (x^4 - y^4) dx \wedge dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
d(\omega \wedge \eta) =&-d(y(x^2 + y^2)) \wedge dy \wedge dz - d(y(x^2 - y^2)) \wedge dz \wedge dx + d(x^4 - y^4) \wedge dx \wedge dy\\
=&-2xy\ dx \wedge dy \wedge dz - (x^2 - 3y^2)\ dy \wedge dz \wedge dx\\
=&(3y^2 - 2xy - x^2) dx \wedge dy \wedge dz.
\end{aligned}
$$

Next, we calculate $d\omega \wedge \eta$. We have:

$$
d\omega = d(x^2 - y^2) \wedge dx + dy \wedge dz
$$

$$\begin{aligned} &= -2y dy \wedge dx + dy \wedge dz \\ &= dy \wedge dz + 2y dx \wedge dy. \end{aligned}$$

Then

$$\begin{aligned} d\omega \wedge \eta =& (dy \wedge dz + 2y dx \wedge dy) \wedge ((x^2 + y^2)\ dy + y\ dz) \\ =& 2y^2 dx \wedge dy \wedge dz. \end{aligned}$$

Finally, we calculate $\omega \wedge d\eta$. We have:

$$\begin{aligned} d\eta =& d(x^2 + y^2) \wedge dy + dy \wedge dz \\ =& 2x\ dx \wedge dy + dy \wedge dz \\ =& dy \wedge dz + 2x\ dx \wedge dy. \end{aligned}$$

Then

$$\begin{aligned} \omega \wedge d\eta =& ((x^2 - y^2)\ dx + y\ dz) \wedge (dy \wedge dz + 2x\ dx \wedge dy) \\ =& (x^2 - y^2) dx \wedge dy \wedge dz + 2xy\ dz \wedge dx \wedge dy \\ =& (x^2 + 2xy - y^2)\ dx \wedge dy \wedge dz. \end{aligned}$$

Putting all this together, we conclude that

$$\begin{aligned} d\omega \wedge \eta - \omega \wedge d\eta =& (3y^2 - 2xy - x^2) dx \wedge dy \wedge dz \\ =& d(\omega \wedge \eta), \end{aligned}$$

as expected from the graded product rule Lemma 4.3.6.

6. Let ω be a k-form, η an m-form, and λ a ℓ-form, all on $U \subset \mathbb{R}^n$. Show that

$$d(\omega \wedge \eta \wedge \lambda) = d\omega \wedge \eta \wedge \lambda + (-1)^k \omega \wedge d\eta \wedge \lambda + (-1)^{k+m} \omega \wedge \eta \wedge d\lambda.$$

Solution. First, using the graded product rule Lemma 4.3.6, we get that

$$d(\omega \wedge \eta \wedge \lambda) = d\omega \wedge (\eta \wedge \lambda) + (-1)^k \omega \wedge d(\eta \wedge \lambda).$$

Next, again from the graded product rule we know that

$$d(\eta \wedge \lambda) = d\eta \wedge \lambda + (-1)^m \eta \wedge d\lambda.$$

Putting this together, we get:

$$d(\omega \wedge \eta \wedge \lambda) = d\omega \wedge \eta \wedge \lambda + (-1)^k \omega \wedge d\eta \wedge \lambda + (-1)^{k+m} \omega \wedge \eta \wedge d\lambda.$$

7. Let $\omega = (x + y)\ dx + xy\ dy$ be a one-form on $\mathbb{R}^2$, and let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\phi(u, v) = (e^{u+v}, e^{u-v})$. Find the two-form $d(\phi^* \omega)$.

Solution. We first calculate the pullback one-form $\phi^* \omega$. We get:

$$\phi^* \omega = (e^{u+v} + e^{u-v}) \left(e^{u+v}\ du + e^{u+v}\ dv \right) + e^{u+v} e^{u-v} \left(e^{u-v}\ du - e^{u-v}\ dv \right)$$

$$= \left(e^{2(u+v)} + e^{2u} + e^{3u-v}\right)\ du + \left(e^{2(u+v)} + e^{2u} - e^{3u-v}\right)\ dv.$$

We can then calculate its exterior derivative. We get:

$$\begin{aligned}
d(\phi^*\omega) =& d\left(e^{2(u+v)} + e^{2u} + e^{3u-v}\right) \wedge du + d\left(e^{2(u+v)} + e^{2u} - e^{3u-v}\right) \wedge dv \\
=& \left(2e^{2(u+v)} - e^{3u-v}\right) dv \wedge du + \left(2e^{2(u+v)} + 2e^{2u} - 3e^{3u-v}\right) du \wedge dv \\
=& 2\left(e^{2u} - e^{3u-v}\right) du \wedge dv.
\end{aligned}$$

8. Let $f, g : U \to \mathbb{R}$ be smooth functions with $U \subseteq \mathbb{R}^n$, and $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve whose image is in U. Using the product rule for the exterior derivative $d(fg)$, show that

$$\int_\alpha f dg = f(\alpha(b))g(\alpha(b)) - f(\alpha(a))g(\alpha(a)) - \int_\alpha g df.$$

One can think of this as the natural generalization of "integration by parts" to line integrals of one-forms. Indeed, if the curve is just an interval in $\mathbb{R}$, this reduces to the standard statement of integration by parts for definite integrals.

Solution. From the graded product rule, we know that

$$d(fg) = f dg + g df,$$

since f is a 0-form (here we omit the wedge product symbol, since we are either multiplying two functions or a function with a one-form). By the Fundamental Theorem of line integrals, we know that

$$\int_\alpha d(fg) = f(\alpha(b))g(\alpha(b)) - f(\alpha(a))g(\alpha(a)),$$

since $d(fg)$ is an exact one-form. Using the graded product rule, we thus conclude that

$$\int_\alpha d(fg) = \int_\alpha f dg + \int_\alpha g df = f(\alpha(b))g(\alpha(b)) - f(\alpha(a))g(\alpha(a)).$$

Solving for $\int_\alpha f dg$, we get the desired statement.

9. Let ω and η be one-forms that differ by the exterior derivative of a 0-form, that is,

$$\eta = \omega + df$$

for some function f. Show that

$$d(\omega \wedge \eta) = d\omega \wedge df.$$

Solution. We have:

$$d(\omega \wedge \eta) = d(\omega \wedge (\omega + df)) = d(\omega \wedge \omega) + d(\omega \wedge df).$$

The first term on the right-hand-side is zero, since $\omega \wedge \omega = 0$ for a one-form (see Lemma 4.2.6; the wedge product is anti-commutative for odd forms). As for the second term, we use the graded product rule and the fact that $d^2 = 0$ to get:

$$d(\omega \wedge \eta) = d\omega \wedge df - \omega \wedge d^2 f = d\omega \wedge df.$$

4.4 The exterior derivative and vector calculus

In this section we continue developing our dictionary between differential forms and standard vector calculus concepts. We introduce the vector calculus operations corresponding to the exterior derivative.

Objectives

You should be able to:

- Relate the exterior derivative of a zero-form in $\mathbb{R}^3$ to the gradient of a function.
- Relate the exterior derivative of a one-form in $\mathbb{R}^3$ to the curl of a vector field.
- Relate the exterior derivative of a two-form in $\mathbb{R}^3$ to the div of a vector field.
- Derive various vector calculus identities from the graded product rule for the exterior derivative and the statement that $d^2 = 0$.

4.4.1 Grad, div and curl

In $\mathbb{R}^3$, we saw that there are three possibilities to get a non-zero differential forms as a result of acting with the exterior derivative: either we take the exterior derivative of a zero-form, a one-form, or a two-form. All three of these operations are given separate names and notation in the standard vector calculus language.

Definition 4.4.1 The gradient of a function. Let f be a zero-form (a function) on $U \subseteq \mathbb{R}^3$. Its exterior derivative df is the one-form

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

We define the **gradient of** f, and denote it by $\boldsymbol{\nabla} f$, to be the vector field associated to the one-form df according to Table 4.1.11:

$$\boldsymbol{\nabla f} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Note that the input of the gradient is a function, and the output is a vector field. ◊

Of course, we already knew the definition of the gradient of a function, and how it is the vector field associated to the exterior derivative of a zero-form: this was already stated in Fact 2.2.2. We include the statement here for completeness. From our point of view, we could take this as the *definition* of the gradient of a function: it is the vector field associated to the exterior derivative of a zero-form.

Definition 4.4.2 The curl of a vector field. Let $\omega = f_1\, dx + f_2\, dy + f_3\, dz$ be a one-form on $U \subseteq \mathbb{R}^3$, with its associated vector field $\mathbf{F} = (f_1, f_2, f_3)$. Its exterior derivative $d\omega$ is the two-form

$$d\omega = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy.$$

We define the **curl of F**, and denote it by $\nabla \times \mathbf{F}$, to be the vector field associated to the two-form $d\omega$ according to Table 4.1.11:

$$\nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right).$$

Note that input of the curl is a vector field, and the output is also a vector field. ◊

Finally, we can apply the exterior derivative to a two-form to get a three-form.

Definition 4.4.3 The divergence of a vector field. Let $\eta = f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_3\, dx \wedge dy$ be a two-form on $U \subseteq \mathbb{R}^3$, with its associated vector field $\mathbf{F} = (f_1, f_2, f_3)$. Its exterior derivative $d\eta$ is the three-form

$$d\eta = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz.$$

We define the **divergence of F**, and denote it by $\nabla \cdot \mathbf{F}$, to be the function associated to the three-form $d\eta$ according to Table 4.1.11:

$$\nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

Note that the input of the diveregence is a vector field, and the output is a function. ◊

Remark 4.4.4 Now you can start to see the power of developing the framework of differential forms. These three operators, namely grad, div, and curl, which appear as independent operators in vector calculus, are just the action of the same operator, namely the exterior derivative, but on zero-, one-, and two-forms respectively. Moreover, we don't need to remember these definitions by heart: all we need to remember is how to act with the exterior derivative on k-forms, which simply amounts to acting with the exterior derivative on the component functions. So much simpler!

Even more powerful is the fact that the framework of differential forms naturally extend to any dimension, not only $\mathbb{R}^3$. However, the defintions of grad, curl, div, the cross-product, etc. rely on the geometry of $\mathbb{R}^3$. The natural generalization to higher dimensions is just the action of the exterior derivative as we defined it.

Remark 4.4.5 The introduction of the curl of a vector field allows us to rephrase the screening test for conservative vector fields in $\mathbb{R}^3$ in a nicer way. Looking at the screening test in Lemma 2.2.16, it is clear that the screening test for a vector field $\mathbf{F}$ is satisfied if and only if

$$\nabla \times \mathbf{F} = 0.$$

In other words, the screening test was simply saying that the curl of the vector field vanishes.

Remark 4.4.6 Just as for the cross product of two vectors, in standard vector calculus textbooks a determinant formula is usually given to remember how to calculate the curl of a vector field $\mathbf{F} = (f_1, f_2, f_3)$ in $\mathbb{R}^3$:

$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the x, y, z directions. You can use this formula if you want. Or you can remember that the curl is obtained by taking the exterior derivative of the one-form associated to $\mathbf{F}$.

Example 4.4.7 Maxwell's equations. Maxwell's equations form the foundations of electromagnetism. It turns out that they are written in terms of the divergence and the curl. More precisely, if $\mathbf{E}$ is the electric vector field, and $\mathbf{B}$ is the magnetic vector field, both defined on $\mathbb{R}^3$ (our space), Maxwell's equations state that

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{E} =& 4\pi\rho,\\ \boldsymbol{\nabla} \cdot \mathbf{B} =& 0,\\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} =& 0,\\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} =& \frac{4\pi}{c}\mathbf{J},\end{aligned}$$

where c is the speed of light, ρ is the total electric charge density, and $\mathbf{J}$ is the total electric current density (which is a vector field). In particular, the equations simplify when there is no charge or current (such as in vacuum), with $\rho = \mathbf{J} = 0$.

Note that we are abusing notation a little bit here. Those equations are the "time-dependent" Maxwell's equations. What this means is that we think of $\mathbf{E}$ and $\mathbf{B}$ as vector fields in $\mathbb{R}^3$ (in space), but that also depend on another variable t corresponding to time. This is why the equations above include partial derivatives of $\mathbf{E}$ and $\mathbf{B}$ with respect to t. The "time-independent" Maxwell's equations, in which $\mathbf{E}$ and $\mathbf{B}$ are true vector fields on $\mathbb{R}^3$ (with no time dependence), would correspond to setting the two terms involving partial derivatives with respect to t to zero. □

We can summarize the dictionary between the exterior derivative in $\mathbb{R}^3$ and vector calculus operations in the following table.

Table 4.4.8 Dictionary between the exterior derivative in $\mathbb{R}^3$ and vector calculus concepts

Differential form concept		Vector calculus concept	
d of a 0-form	df	gradient	$\boldsymbol{\nabla} f$
d of a 1-form	$d\omega$	curl	$\boldsymbol{\nabla} \times \mathbf{F}$
d of a 2-form	$d\eta$	divergence	$\boldsymbol{\nabla} \cdot \mathbf{F}$

4.4.2 The graded product rule and vector calculus identities

The power of the formalism of differential forms is further highlighted by the following lemma. We saw above that the standard concepts of grad, curl, and div, are just reformulations of the exterior derivative. We showed in Lemma 4.3.6 that the exterior derivative satisfies a graded product rule. In $\mathbb{R}^3$, this graded product rule can be split into two cases, as in Remark 4.3.8, depending on whether ω is a zero- or a one-form. Using the definition of grad, curl, and div in Definition 4.4.1, Definition 4.4.2 and Definition 4.4.3, those statements can be translated into vector calculus identities. The result is the following lemma.

Lemma 4.4.9 Vector calculus identities, part 1. *Let f, g be smooth functions on $U \subseteq \mathbb{R}^3$, and $\mathbf{F}, \mathbf{G}$ be smooth vector fields on U. Then the following identities are satisfied:*

1. $$\boldsymbol{\nabla}(fg) = (\boldsymbol{\nabla} f)g + f(\boldsymbol{\nabla} g),$$

2. $$\boldsymbol{\nabla} \times (f\mathbf{F}) = (\boldsymbol{\nabla} f) \times \mathbf{F} + f\boldsymbol{\nabla} \times \mathbf{F},$$

3. $$\boldsymbol{\nabla} \cdot (f\mathbf{F}) = (\boldsymbol{\nabla} f) \cdot \mathbf{F} + f\boldsymbol{\nabla} \cdot \mathbf{F},$$

4. $$\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}).$$

Proof. This is just the reformulation in terms of vector fields of the four different non-vanishing cases of the graded product rule for differential forms in $\mathbb{R}^3$ (see Remark 4.3.8). ■

Now you may be starting to like this. Learning this kind of vector calculus identities by heart is frustrating. But these are just reformulations of the one and only graded product rule for the exterior derivative Lemma 4.3.6, which is all that you have to remember (sure, there is an annoying sign in the graded product rule, but it's much better than learning vector calculus identities!).

4.4.3 $d^2 = 0$ and vector calculus identities

Another key property of the exterior derivative is that $d^2 = 0$, see Lemma 4.3.9. In $\mathbb{R}^3$, this corresponds to two separate statements, namely that $d(d(f)) = 0$ with f a zero-form (a function), and $d(d(\omega)) = 0$ with ω a one-form. The corresponding vector calculus identities are the following:

Lemma 4.4.10 Vector calculus identities, part 2. *Let f be a smooth function on $U \subseteq \mathbb{R}^3$ and $\mathbf{F}$ a smooth vector field on U. Then the following identities are satisfied:*

1. $$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0 \qquad \textit{(curl of grad is zero)},$$

2. $$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0 \qquad \textit{(div of curl is zero)}.$$

Proof. This is just the reformulation of the statement that $d^2 = 0$ for a zero-form and a one-form in $\mathbb{R}^3$. ■

The game of translating easy statements for the exterior derivative into complicated statements for grad, curl, and div is fun, isn't it? Let's prove one more vector calculus identity for now, which follows by combining the statement that $d^2 = 0$ with the graded product rule.

Lemma 4.4.11 Vector calculus identities, part 3. *Let f, g, h be smooth functions on $U \subseteq \mathbb{R}^3$. Then:*

$$\boldsymbol{\nabla} \cdot (f(\boldsymbol{\nabla} g \times \boldsymbol{\nabla} h)) = \boldsymbol{\nabla} f \cdot (\boldsymbol{\nabla} g \times \boldsymbol{\nabla} h).$$

Proof. Consider the action of the exterior derivative on the two-form $f dg \wedge dh$:

$$d(f dg \wedge dh) = df \wedge (dg \wedge dh) + f d(dg \wedge dh),$$

where we used the graded product rule and the fact that f is a zero-form. Using the graded product rule again, we know that

$$\begin{aligned} d(dg \wedge dh) =& d(dg) \wedge dh - dg \wedge d(dh) \\ =& 0, \end{aligned}$$

since $d(dg) = d(dh) = 0$. Thus

$$d(f dg \wedge dh) = df \wedge (dg \wedge dh).$$

The translation for the associated vector fields is:

$$\boldsymbol{\nabla} \cdot (f(\boldsymbol{\nabla g} \times \boldsymbol{\nabla} h)) = \boldsymbol{\nabla} f \cdot (\boldsymbol{\nabla} g \times \boldsymbol{\nabla} h),$$

as claimed. ■

4.4.4 Two more vector calculus identities

We end this section by noting that there are two more vector calculus identities involving grad, curl and div. We will present the identities without proof here. To get them from differentials forms, we would need to introduce the concept of Lie derivatives, which is beyond the scope of this course.

Keep in mind that you certainly do not need to learn these identities by heart! We are presenting them here just so that you are aware of them.

Lemma 4.4.12 Vector calculus identities, part 4. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be smooth vector fields on* $U \subseteq \mathbb{R}^3$*. Then the following identities are satisfied:*

1. $$\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + (\boldsymbol{\nabla} \times \mathbf{F}) \times \mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G},$$

2. $$\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - (\boldsymbol{\nabla} \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}.$$

Here $(\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F}$ *means*

$$(\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} = G_1 \frac{\partial}{\partial x}\mathbf{F} + G_2 \frac{\partial}{\partial y}\mathbf{F} + G_3 \frac{\partial}{\partial z}\mathbf{F}.$$

Finally, there are a few more vector calculus identities that involve the Laplacian operator, which in the language of differential forms requires the introduction of the Hodge star operator. We will come back to this in Section 4.8.

4.4.5 Exercises

1. Let $\mathbf{F} = (xy, yz, xz)$ be a vector field on $\mathbb{R}^3$. Find its curl $\boldsymbol{\nabla} \times \mathbf{F}$ and divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$.

Solution. The curl of the vector field is given by:

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} =& \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{pmatrix} \\ =& -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.\end{aligned}$$

In component notation, this reads $\boldsymbol{\nabla} \times \mathbf{F} = (-y, -z, -x)$.

As for the div, we get:

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{F} =& \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) \\ =& y + z + x.\end{aligned}$$

Note that we could have done these calculations using differential forms. To get the curl, we associate to $\mathbf{F}$ a one-form $\omega = xy\ dx + yz\ dy + xz\ dz$ and calculate its exterior derivative:

$$\begin{aligned}d\omega =& xdy \wedge dx + ydz \wedge dy + zdx \wedge dz \\ =& -ydy \wedge dz - zdz \wedge dx - xdx \wedge dy\end{aligned}$$

The curl $\boldsymbol{\nabla} \times \mathbf{F}$ is then the vector field associated to this two-form, that is $\boldsymbol{\nabla} \times \mathbf{F} = (-y, -z, -x)$.

To calculate the divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$, we associate to $\mathbf{F}$ a two-form $\eta = xy\ dy \wedge dz + yz\ dz \wedge dx + xz\ dx \wedge dy$ and calculate its exterior derivative:

$$\begin{aligned}d\eta =& ydx \wedge dy \wedge dz + zdy \wedge dz \wedge dx + xdz \wedge dx \wedge dy \\ =& (x + y + z)dx \wedge dy \wedge dz.\end{aligned}$$

Therefore $\boldsymbol{\nabla} \cdot \mathbf{F} = x + y + z$.

2. For the following two vector fields, find their curl and divergence:

(a) $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2}}(x, y, 0)$.

(b) $\mathbf{G}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2}}(-y, x, 0)$.

Solution. Let's solve this one using differential forms. You can do it directly using the formulae for curl and div as well.

(a) To find the curl, we associated a one-form ω to $\mathbf{F}$:

$$\omega = \frac{1}{\sqrt{x^2 + y^2}}(x\ dx + y\ dy).$$

We calculate its exterior derivative:

$$\begin{aligned}d\omega =& -\frac{xy}{(x^2+y^2)^{3/2}}dy\wedge dx - \frac{xy}{(x^2+y^2)^{3/2}}dx\wedge dy\\ =&0.\end{aligned}$$

Thus we conclude that $\mathbf{\nabla}\times\mathbf{F}=0$.

To find the divergence, we associate a two-form η to $\mathbf{F}$:

$$\eta = \frac{1}{\sqrt{x^2+y^2}}(x\ dy\wedge dz + y\ dz\wedge dx).$$

We calulate its exterior derivative:

$$\begin{aligned}d\eta =& \left((x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}\right)dx\wedge dy\wedge dz\\ &+\left((x^2+y^2)^{-1/2} - y^2(x^2+y^2)^{-3/2}\right)dy\wedge dz\wedge dx\\ =&\left(\frac{2}{\sqrt{x^2+y^2}} - \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right)dx\wedge dy\wedge dz\\ =&\frac{1}{\sqrt{x^2+y^2}}dx\wedge dy\wedge dz.\end{aligned}$$

We conclude that

$$\mathbf{\nabla}\cdot\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}}.$$

(b) To find the curl, we associate a one-form ω to $\mathbf{G}$:

$$\omega = \frac{1}{\sqrt{x^2+y^2}}(-y\ dx + x\ dy).$$

We calculate its exterior derivative:

$$\begin{aligned}d\omega =& \left(-(x^2+y^2)^{-1/2} + y^2(x^2+y^2)^{-3/2}\right)dy\wedge dx\\ &+\left((x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}\right)dx\wedge dy\\ =&\left(\frac{2}{\sqrt{x^2+y^2}} - \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right)dx\wedge dy\\ =&\frac{1}{\sqrt{x^2+y^2}}dx\wedge dy.\end{aligned}$$

Thus we conclude that

$$\mathbf{\nabla}\times\mathbf{G} = \left(0, 0, \frac{1}{\sqrt{x^2+y^2}}\right).$$

To find the divergence, we associate to $\mathbf{G}$ the two-form:

$$\eta = \frac{1}{\sqrt{x^2+y^2}}(-y\ dy\wedge dz + x\ dz\wedge dx).$$

We calculate its exterior derivative:

$$\begin{aligned}d\eta =&\frac{xy}{(x^2+y^2)^{3/2}}dx\wedge dy\wedge dz - \frac{xy}{(x^2+y^2)^{3/2}}dy\wedge dz\wedge dx\\ =&0\end{aligned}$$

Therefore $\mathbf{\nabla}\cdot\mathbf{G}=0$.

3. Find a vector field $\mathbf{F} = (0, f_2, f_3)$ such that $\boldsymbol{\nabla} \times \mathbf{F} = (0, z, y)$.

Solution. Since $\mathbf{F} = (0, f_2, f_3)$, and using the definition of the curl, we know that

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, -\frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} \right).$$

Thus we need to solve the equations

$$\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = 0, \qquad -\frac{\partial f_3}{\partial x} = z, \qquad \frac{\partial f_2}{\partial x} = y.$$

Integrating the last two equations, we get:

$$f_2 = xy + g(y, z), \qquad f_3 = -xz + h(y, z),$$

for some functions $g(y, z), h(y, z)$. The first condition then imposes that

$$\frac{\partial}{\partial y} h(y, z) = \frac{\partial}{\partial z} g(y, z).$$

There are many possible choices, but the simplest would be $g(y, z) = h(y, z) = 0$. We would then conlude that

$$\mathbf{F} = (0, xy, -xz)$$

is a vector field such that $\boldsymbol{\nabla} \times \mathbf{F} = (0, z, y)$.

4. Is there a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\boldsymbol{\nabla} \times \mathbf{F} = (x, y + xz, z)$? Justify your answer.

Solution. We know that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$. So if there is a vector field $\mathbf{F}$ such that $\boldsymbol{\nabla} \times \mathbf{F} = (x, y + xz, z)$, then the divergence of the vector on the right-hand-side (let's call it $\mathbf{G}$) must be zero. But

$$\boldsymbol{\nabla} \cdot \mathbf{G} = 1 + 1 + 1 = 3,$$

which is obviously non-zero. Therefore, we conclude that there does not exist a vector field $\mathbf{F}$ such that $\boldsymbol{\nabla} \times \mathbf{F} = (x, y + xz, z)$.

5. Let $\mathbf{F} = (xy, y^2, xy + z)$ and $\mathbf{G} = (xyz, yz, z^2)$ be smooth vector fields on $\mathbb{R}^3$, and $\alpha : [0, 2\pi] \to \mathbb{R}^3$ be the parametric curve given by $\alpha(t) = (\sin(t), \cos(t), t(t - 2\pi))$. Show that the line integral of the vector field

$$\mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + (\boldsymbol{\nabla} \times \mathbf{F}) \times \mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}$$

along α is zero.

Solution. Well, you certainly do not want to evaluate this line integral, it would be painful!

First, we notice that the parametric curve α is closed, since

$$\alpha(0) = (0, 1, 0) = \alpha(2\pi).$$

Next, we notice that we can use the vector calculus identity 1 from Lemma 4.4.12, which states that

$$\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + (\boldsymbol{\nabla} \times \mathbf{F}) \times \mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}.$$

So the vector field that we want to evaluate the line integral of is $\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G})$. As this is the gradient of a function, this means that the vector field is conservative. Therefore, its integral along any closed curve vanishes. We conclude that the integral along α is zero!

6. Suppose that you study a vector field $\mathbf{F}$ in a lab. You measure that

$$\mathbf{F}(x,y,z) = (xz + yz + x^2y, \alpha(yz + x^2z), \beta(xyz + y)),$$

for some constants α, β that you are not able to determine experimentally. However, from theoretical considerations you know that $\mathbf{F}$ must be divergence-free (i.e., its divergence is zero). Find the values of α and β.

Solution. We know that

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} =& \frac{\partial}{\partial x}(xz + yz + x^2y) + \alpha\frac{\partial}{\partial y}(yz + x^2z) + \beta\frac{\partial}{\partial z}(xyz + y)\\ =& z + 2xy + \alpha z + \beta xy.\end{aligned}$$

Since we know that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$, and that α and β are constants (i.e. do not depend on x, y, z), we conclude that we must have

$$\alpha = -1, \qquad \beta = -2.$$

7. Let $\mathbf{F}(x,y,z) = (f(x), g(y), h(z))$ for some smooth functions $f(x), g(y), h(z)$ on $\mathbb{R}$ (note that those are functions of a single variable), and let $q(x,y,z)$ be an arbitrary smooth function on $\mathbb{R}^3$. Show that

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\boldsymbol{\nabla}q) = 0.$$

Solution. Let us first solve it using vector calculus identities, and then provide an alternative but equivalent solutions using differential forms. Identity 4 of Lemma 4.4.9 states that

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\mathbf{G}) = (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\boldsymbol{\nabla}\times\mathbf{G}).$$

Applying this to the case at hand, we get:

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\boldsymbol{\nabla}q) = (\boldsymbol{\nabla}\times\mathbf{F})\cdot\boldsymbol{\nabla}q - \mathbf{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}q).$$

We then calculate the curl of $\mathbf{F}$. We get:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} =& \left(\frac{\partial h(z)}{\partial y} - \frac{\partial g(y)}{\partial z}, \frac{\partial f(x)}{\partial z} - \frac{\partial h(z)}{\partial x}, \frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\right)\\ =& 0.\end{aligned}$$

Moreover, from Identity 1 of Lemma 4.4.10, we know that

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}q = 0.$$

Therefore

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\boldsymbol{\nabla}q) = 0.$$

Let us now solve the question using differential forms. Let $\omega = f(x)\ dx + g(y)\ dy + h(z)\ dz$ be the one-form associated to $\mathbf{F}$. Then $\boldsymbol{\nabla}\cdot(\mathbf{F}\times\boldsymbol{\nabla}q)$ is the function associated to $d(\omega\wedge dq)$. So we want to show that

$$d(\omega\wedge dq) = 0.$$

Using the graded product rule, we have:

$$\begin{aligned} d(\omega \wedge dq) =& d\omega \wedge dq - \omega \wedge d^2 q \\ =& d\omega \wedge dq, \end{aligned}$$

where we used the fact that $d^2 = 0$. But

$$\begin{aligned} d\omega =& df(x) \wedge dx + dg(y) \wedge dy + dh(z) \wedge dz \\ =& 0, \end{aligned}$$

since by evaluating the differentials we only get terms involving the vanishing basic two-forms $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. We thus conclude that

$$d(\omega \wedge dq) = 0.$$

4.5 Physical interpretation of grad, curl, div

In this section we pause for a moment and explore further the physical interpretation of the vector calculus operations of gradient, curl, and divergence.

Objectives

You should be able to:

- Interpret the gradient of a vector field as giving the direction and magnitude of fastest increase.
- Interpret the curl of a vector field in terms of a rotational motion in a fluid.
- Interpret the divergence of a vector field in terms of expansion and contraction of a fluid.

4.5.1 The gradient of a function

Let $f : U \to \mathbb{R}$ be a smooth function with $U \subseteq \mathbb{R}^3$. We start by reviewing the interpretation of the gradient

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

which as we saw is the vector field associated to the exterior derivative df. At a point $p \in U$:

- The direction of the gradient is the direction in which the function f increases most quickly at p;
- The magnitude of the gradient is the rate of fastest increase at p.

This may be most easily understood with an example.

Example 4.5.1 The direction of steepest slope. In this example we work with a function on $\mathbb{R}^2$ instead of $\mathbb{R}^3$, but the interpretation is the same.

Suppose that the function $H(x, y) = -x^2 - y^2 + 1000$ gives the height above sea level at point (x, y) on a surface. The height is highest at the origin, where $H(0, 0) = 1000$, and decreases as we move away from the origin, until it reaches sea level on the circle with radius $\sqrt{1000}$. So we can think of this altitude function as representing in a circular mountain centred at the origin.

The level curves of H are the curves of constant elevation, i.e.

$$H(x,y) = -x^2 - y^2 + 1000 = C \qquad \Leftrightarrow \qquad x^2 + y^2 = 1000 - C$$

for constant elevations $0 \leq C \leq 1000$. Those would correspond to the level curves of elevation on a topographical map. In this case, they are all circles centred at the origin.

The gradient of H is

$$\boldsymbol{\nabla} H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right) = -2(x,y).$$

For any point (x,y) on the surface, this is a vector that points towards the origin, which says that the direction of steepest increasing slope is towards the origin, as expected from a circular mountain centred at the origin. Moreover, we see that the magnitude of the gradient vector is

$$|\boldsymbol{\nabla} H| = 2\sqrt{x^2+y^2}.$$

This says that the slope is steeper far away from the origin, and becomes less and less steep as we get closer to the origin.

Here is below the contour map for this function (the graph of its level curves), and also a 3D plot (both produced using Mathematica).

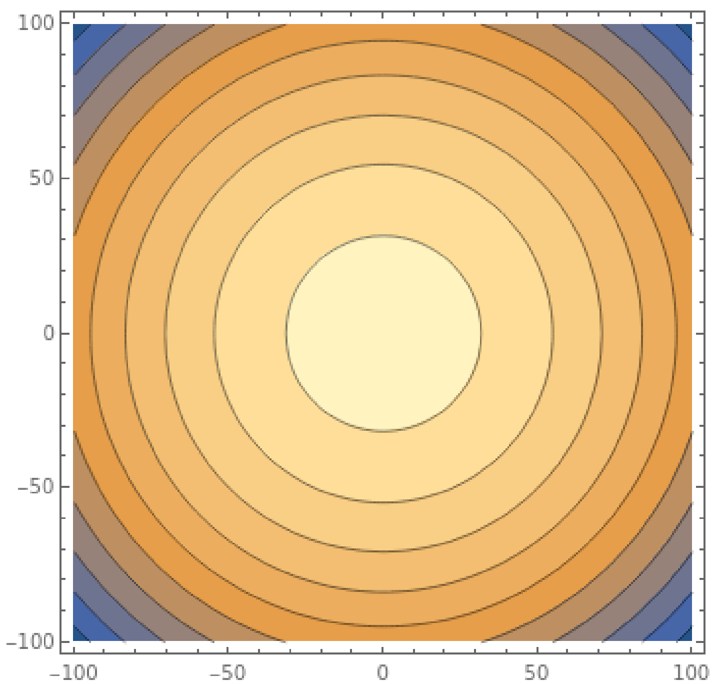


Figure 4.5.2 A contour map of the function $H(x,y) = -x^2 - y^2 + 1000$, for $-100 \leq x \leq 100$ and $-100 \leq y \leq 100$.

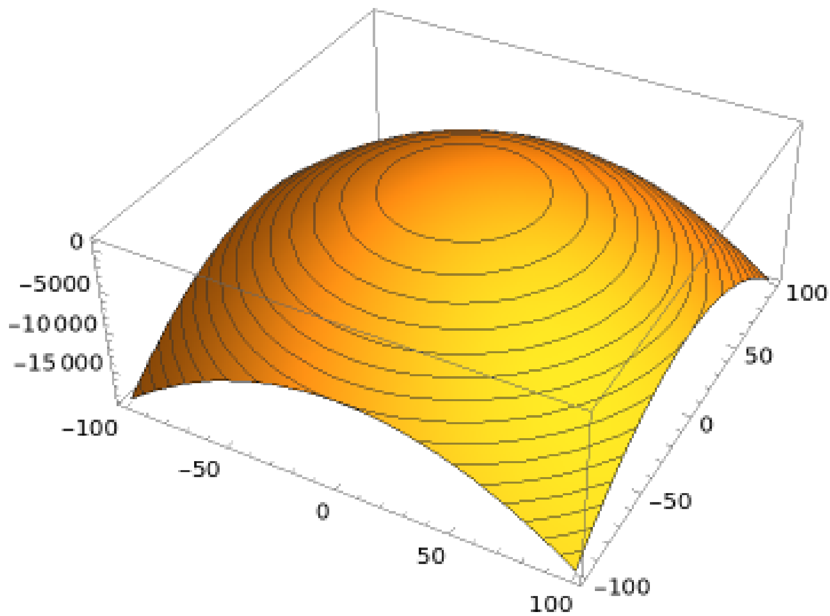


Figure 4.5.3 A 3d plot of the function $z = -x^2 - y^2 + 1000$ (i.e. thinking of $H(x, y)$ as representing the height of the surface above sea level), for $-100 \leq x \leq 100$ and $-100 \leq y \leq 100$.

□

4.5.2 The curl of a vector field

Let $\mathbf{F} : U \to \mathbb{R}^3$ be a smooth vector field with $U \subseteq \mathbb{R}^3$, with component functions $\mathbf{F} = (f_1, f_2, f_3)$. Its curl is given by the vector field

$$\nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right).$$

What is it actually computing? How can we interpret the vector field $\nabla \times \mathbf{F}$?

It is easiest to understand the meaning of the curl of a vector field by thinking of $\mathbf{F}(x, y, z)$ as the velocity field of a moving fluid in three dimensions. So let us call it $\mathbf{v}(x, y, z)$ instead. The curl is related to rotational motion induced by the fluid.

Consider an infinitesimally small ball (sphere) located within the fluid, and centered at a point $p \in U$. Assume that the ball has a rough surface. The fluid moving around the ball will generally make it rotate. Then:

- The direction of the curl $\nabla \times \mathbf{v}$ at p gives the axis of rotation (according to the right hand rule);[1]
- Half the magnitude of the curl $\nabla \times \mathbf{v}$ at p gives the angular speed of rotation.

In particular, if $\nabla \times \mathbf{v} = 0$ at a point p, the fluid does not cause the sphere centered at p to rotate. Because of this interpretation, we say that the velocity field of a fluid is **irrotational** if it is curl-free at all points on U, that is $\nabla \times \mathbf{v} = 0$ on U.

[1]The “right hand rule” means that if you align the thumb of your right hand along the vector, the fingers of your right hand will “curl” around the axis of rotation in the direction of rotation.

Example 4.5.4 The curl of the velocity field of a moving fluid. Suppose that a moving fluid has velocity field given by

$$\mathbf{v}(x, y, z) = (-y, x, 0).$$

Sketching the vector field (see Exercise 2.1.3.2), one sees that at any point, the fluid is rotating counterclockwise around the z-axis. Using the right-hand-rule, we thus expect the curl $\boldsymbol{\nabla} \times \mathbf{v}$ to point in the positive z-direction, since it is the axis of rotation. We calculate:

$$\boldsymbol{\nabla} \times \mathbf{v} = (0, 0, 2)\,.$$

Indeed, it points in the positive z-direction, as expected. Furthermore, half the magnitude of the curl is

$$\frac{1}{2}|\boldsymbol{\nabla} \times \mathbf{v}| = 1,$$

which means that the angular speed of rotation of the ball would be 1 radian per unit of time. □

Example 4.5.5 An irrotational velocity field. Conside a moving fluid with velocity field given by

$$\mathbf{v}(x, y, z) = (x, y, z).$$

Sketching the vector field, we see that this would be a fluid in expansion. As the fluid is expanding, regardless of where the small sphere is located, it should not cause it to rotate (try to visualize this yourself). So we expect the velocity field to be curl-free. From the definition of the curl, we calculate:

$$\boldsymbol{\nabla} \times \mathbf{v} = (0, 0, 0).$$

Thus it is curl-free, as expected, and this is an example of an irrotational velocity field. □

Example 4.5.6 Another irrotational velocity field. Irrotational fluids do not have to be necessarily spherically symmetric. Consider for instance a moving fluid with velocity field

$$\mathbf{v}(x, y, z) = (0, 0, 1).$$

This would be a fluid that is moving uniformly in the positive z-direction. If you think about it for a little bit, it should not induce any rotation either, regardless of where the small sphere is located. So we expect the velocity field to be curl-free again. Indeed, from the definition we $\boldsymbol{\nabla} \times \mathbf{v} = (0, 0, 0)$, which says that the velocity field is irrotational. □

4.5.3 The divergence of a vector field

Let $\mathbf{F} : U \to \mathbb{R}^3$ be a smooth vector field with $U \subseteq \mathbb{R}^3$, with component functions $\mathbf{F} = (f_1, f_2, f_3)$. Its divergence is the function:

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

What is its intepretation?

As for the curl, it is easiest to interpret the divergence by thinking of the vector field as being the velocity field $\mathbf{v}(x, y, z)$ of a moving fluid or gas. The divergence is then related to expansion and compression of the fluid.

More precisely, consider an infinitesimally small sphere around a point $p \in U$:

- The divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$ at p measures the rate at which the fluid is exiting the small sphere at p (per unit of time and unit of volume).

The key here is that if the divergence at a point is positive, it means that more fluid is exiting the small sphere centered at this point then entering the sphere, and vice-versa if the divergence is negative. If it is zero, this means that the fluid may still be moving, but there is exactly the same amount of fluid entering and existing the sphere.

For instance, if a gas is heated, it will expand. This means that for any small sphere in the gas, there will be more gas exiting then entering. So the divergence will be positive everywhere. Conversely, if a gas is cooled, it will contract, and its divergence will be negative everywhere.

Motivated by this interpretation, we say that the velocity field of a fluid is **incompressible** if it is divergence-free everywhere, i.e. $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere on U. We also call such vector fields **solenoidal**.

Example 4.5.7 The divergence of the velocity field of an expanding fluid. Consider a fluid/gas with velocity field

$$\mathbf{v}(x, y, z) = (x, y, z).$$

The velocity field is pointing outwards in all directions. This corresponds to an expanding fluid. If we first think of a small sphere centered at the origin, then the fluid is moving outwards in all directions, exiting the sphere, and thus we expect the divergence to be positive. In fact, even if the small sphere is located elsewhere, we still expect the divergence to be positive, as there will be more fluid exiting the sphere than entering the sphere. From the definition, we calculate

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 1 + 1 + 1 = 3,$$

which is indeed positive everywhere, as expected. □

Example 4.5.8 An imcompressible velocity field. Consider a fluid with velocity field

$$\mathbf{v}(x, y, z) = (-y, x, 0).$$

As we have seen (see Exercise 2.1.3.2), the fluid is rotating counterclockwise around the z-axis. It is not so obvious to see whether more fluid is existing or entering small spheres in the fluid.

It is easiest to consider first a sphere centered around the origin. Because of the rotational motion, we see that actually no fluid is entering or leaving the sphere at all. So we expect the divergence to be zero, at least at the origin.

It is not so obvious to see why the same should be true for all spheres not centered at the origin, but you can try to visualize it. In the end, through direct calculation, we get that $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, and hence the velocity field is divergence-free (that is, incompressible). □

Example 4.5.9 Another incompressible velocity field. Consider the fluid with velocity field

$$\mathbf{v}(x, y, z) = (0, 0, 1),$$

which has uniform velocity in the positive z-direction. Here, if you pick a small sphere centered anywhere, there is certainly fluid entering and exiting the sphere because of the linear motion of the fluid, but the exact same amount of fluid will enter and exit the sphere. Therefore we expect the divergence to be zero everywhere, and indeed, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. Another example of an incompressible velocity field.

Comparing with Example 4.5.6, we see that this velocity field is both irrotational and incompressible, since it is both curl-free and divergence-free. □

4.5.4 Exercises

1. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$$

is irrotational.

Solution. To show that $\mathbf{F}$ is irrotational, we need to show that

$$\boldsymbol{\nabla} \times \mathbf{F} = 0.$$

We can use directly the formula for the curl, or we can use the language of differential forms. In the latter, we associate a one-form ω to the vector field $\mathbf{F}$:

$$\omega = f(x)\ dx + g(y)\ dy + h(z)\ dz.$$

We want to show that $d\omega = 0$. But this is clear true, as taking the exterior derivative gives

$$d\omega = \frac{df}{dx}\ dx \wedge dx + \frac{dg}{dy}\ dy \wedge dy + \frac{dh}{dz}\ dz \wedge dz = 0,$$

since $f(x), g(y), h(z)$ are only functions of x, y, z respectively. Therefore $\mathbf{F}$ is irrotational.

2. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$$

is incompressible.

Solution. To show that $\mathbf{F}$ is incompressible, we need to show that

$$\boldsymbol{\nabla} \cdot \mathbf{F} = 0.$$

This is the statement that

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0.$$

But each of those partial derivatives vanishes, since $f(y, z)$ does not depend on x, $g(x, z)$ does not depend on y, and $h(x, y)$ does not depend on z. Therefore $\mathbf{F}$ is incompressible.

3. Consider a vector field $\mathbf{F}(x, y, z) = (f_1(x, y), f_2(x, y), 0)$ on $\mathbb{R}^3$; it is independendent of z, and its z-component is zero. A sketch of the vector field in the xy-plane is shown in the figure below; as it is independent of z, it looks the same in all other horizontal planes.

(a) Is $\boldsymbol{\nabla} \cdot \mathbf{F}$ positive, negative, or zero at the origin? What about at the point $(5, 0, 0)$? And $(-5, 0, 0)$?

(b) Is $\nabla \times \mathbf{F} = 0$ at the origin? If not, in what direction does it point?

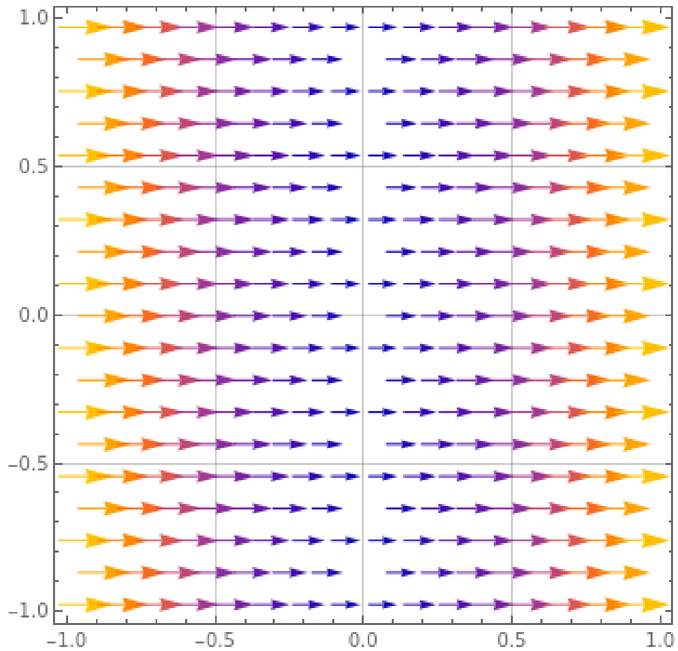


Figure 4.5.10 A sketch of the vector field $\mathbf{F}$ in the xy-plane.

Solution. (a) First we see that all arrows point in the positive x-direction. The lengths of the arrows appear to be symmetric about the y-axis. The arrows get longer and longer as you move away from the y-axis.

If you picture a small sphere around the origin, then there is fluid entering and leaving the sphere, and the arrows of the vectors entering and exiting have the same length. So you expect the divergence to be zero at the origin. In fact, the same is true for all points on the y-axis, i.e. with $x = 0$.

At the point $(5, 0, 0)$, there are arrows entering and leaving the sphere as well. But the arrows that leave the sphere are longer than those that enter, so there should be more fluid exiting than entering. So we expect the divergence to be positive. This should be the case for all points (x, y) with $x > 0$.

As for $(-5, 0, 0)$, the opposite happens; the arrows entering the sphere are longer than those leaving. So there should be more fluid entering than leaving, and we expect the divergence to be negative. The same is true for all points (x, y) with $x < 0$.

(b) At the origin, the arrows are all pointing horizontally. As such, it will not induce any rotation on a sphere centered at the origin, and we expect the curl to be zero at the origin. In fact, this will be the case everywhere, so the curl should be zero everywhere.

For your interest, this is the vector field $\mathbf{F}(x, y, z) = (x^2, 0, 0)$. Its divergence is $\nabla \cdot \mathbf{F} = 2x$. We see that it is zero when $x = 0$, positive for all $x > 0$, and negative for all $x < 0$, as expected. The curl is $\nabla \times \mathbf{F} = (0, 0, 0)$, which vanishes as expected (the vector field is irrotational).

4. Consider a vector field $\mathbf{F}(x, y, z) = (f_1(x, y), f_2(x, y), 0)$ on $\mathbb{R}^3$; it is independent of z, and its z-component is zero. A sketch of the vector field in the xy-plane is shown in the figure below; as it is independent of z, it looks the same in all other horizontal planes.

(a) Is $\nabla \cdot \mathbf{F}$ positive, negative, or zero at the origin? What about at $(0,5,0)$?

(b) Is $\nabla \times \mathbf{F} = 0$ at the origin? If not, in what direction does it point?

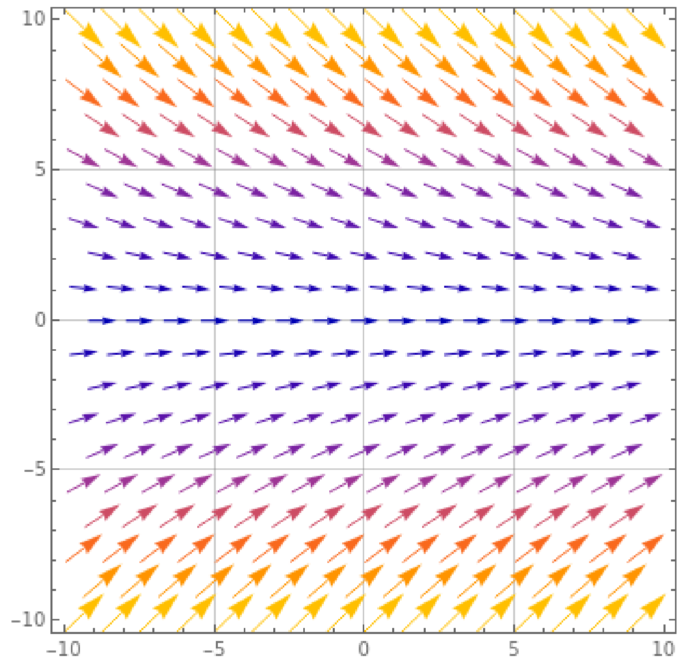


Figure 4.5.11 A sketch of the vector field $\mathbf{F}$ in the xy-plane.

Solution. (a) We look at a small sphere around the origin, and we want to know whether there is more or less fluid entering the sphere versus exiting the sphere. Around the origin, we see that the arrows appear to be the same length on both sides of the y-axis, and on both sides of the x-axis. The arrows are all pointing the positive x-direction. Furthermore, the arrows above the x-axis point downwards, while the arrows below the x-axis point upwards, in a symmetric way. If you think about it, this means that there is more fluid entering the sphere than exiting (because the arrows above and below the x-axis are pointing towards the x-axis). We thus expect the divergence to be negative at the origin.

At the point $(0,5,0)$, the fluid is moving downwards, and the arrows appear to be bigger for $y > 5$ than for $y < 5$. So it looks like there is still more fluid entering the sphere than exiting the sphere. So we expect the divergence to be negative at this point as well.

(b) At the origin, all arrows point in the positive x-direction. Furthermore, the arrows above the x-axis point downwards, while the arrows below the x-axis point upwards, in a symmetric way. Therefore, if you imagine a small sphere around the origin, the motion of the fluid would not induce any rotation. So we expect its curl to vanish.

For your interest, this is the vector field $\mathbf{F}(x,y,z) = (10, -y, 0)$. Its divergence is $\nabla \cdot \mathbf{F} = -1$, which is negative everywhere. As for the curl, we get $\nabla \times \mathbf{F} = (0,0,0)$, so it is zero everywhere (the vector field is irrotational).

5. Consider a vector field $\mathbf{F}(x, y, z) = (f_1(x, y), f_2(x, y), 0)$ on $\mathbb{R}^3$; it is independendent of z, and its z-component is zero. A sketch of the vector field in the xy-plane is shown in the figure below; as it is independent of z, it looks the same in all other horizontal planes.

(a) Is $\boldsymbol{\nabla} \cdot \mathbf{F}$ positive, negative, or zero at the origin? What about at $(0, 5, 0)$?

(b) Is $\boldsymbol{\nabla} \times \mathbf{F} = 0$ at the origin? If not, in what direction does it point?

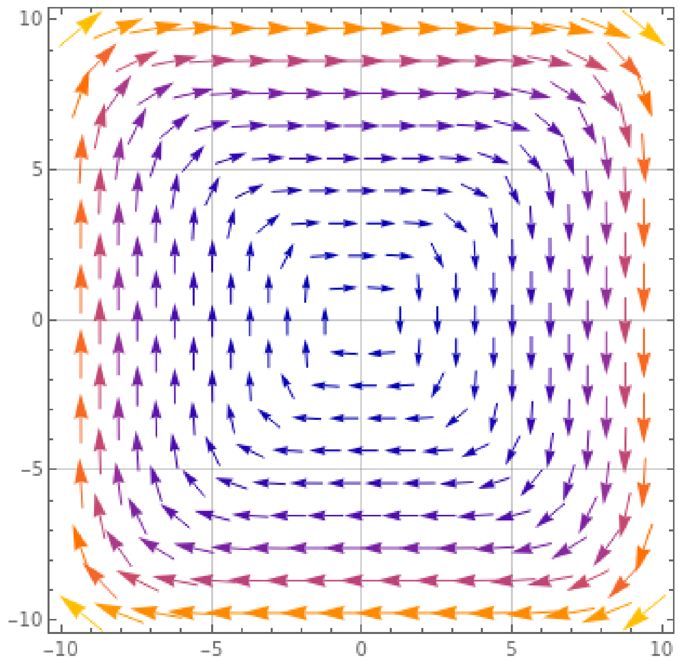


Figure 4.5.12 A sketch of the vector field $\mathbf{F}$ in the xy-plane.

Solution. (a) Arrows that are diametrically opposite around the origin appear to have the same length but point in opposite directions. As a result, the same amount of fluid should be entering and exiting a small sphere around the origin, and hence we expect the divergence to be zero at the origin.

As for the point $(0, 5, 0)$, we see that at this point (just like for any point with $x = 0$) all arrows are pointing horizontally. Furthermore, the length of the arrows on both sides of the y-axis appear to be the same, so we expect the same amount of fluid entering and exiting the sphere. Again, the divergence should be zero at this point.

(b) It is clear from the figure that the moving fluid would make a sphere centered at the origin rotate clockwise. We thus expect the curl to be non-zero at the origin. Using the right hand rule, we expect it to point in the negative z-direction. In fact, looking at the figure, we expect this to be true at all points. So we expect the divergence to be zero everywhere.

For your interest, this is the vector field $\mathbf{F}(x, y, z) = (y^5, -x^5, 0)$. Its divergence is $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$,, which is zero everywhere, as expected (the vector field is incompressible). Its curl is $\boldsymbol{\nabla} \times \mathbf{F} = (0, 0, -5x^4 - 5y^4)$. We see that it points in the negative z-direction for all values of x, y, as expected.

6. Consider a vector field $\mathbf{F}(x, y, z) = (f_1(x, y), f_2(x, y), 0)$ on $\mathbb{R}^3$; it is independendent of z, and its z-component is zero. A sketch of the vector field in the xy-plane is shown in the figure below; as it is independent of z, it looks the same in all other horizontal planes.

(a) Is $\boldsymbol{\nabla} \cdot \mathbf{F}$ positive, negative, or zero at the origin? What about at $(0, 5, 0)$?

(b) Is $\boldsymbol{\nabla} \times \mathbf{F} = 0$ at the origin? If not, in what direction does it point?

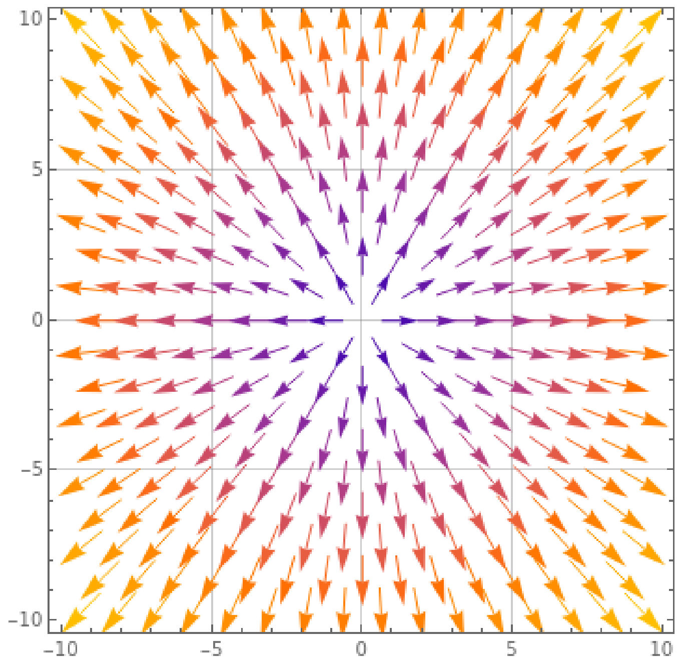


Figure 4.5.13 A sketch of the vector field $\mathbf{F}$ in the xy-plane.

Solution. (a) We see that all arrows point away from the origin, in a way that is spherically symmetric (the lengths of all arrows on a circle of a fixed radius about the origin appear to be the same). As such, the fluid is all exiting the sphere, so we expect the divergence to be positive at the origin.

At the point $(0, 5, 0)$, there are arrows coming in and arrows coming out, but the arrows coming out are longer than the arrows coming out, so we expect again the divergence to be positive since more fluid is exiting the sphere than entering. In fact, this will be the case at all points in the figure. So we expect the divergence to be always positive.

(b) At the origin, the fluid is all pushing outwards in spherically symmetric way, so it will induce no rotation on a sphere centered at the origin. We thus expect the curl to be zero at the origin. While it may not be as obvious, you can probably convince yourself that this should be true at all points in the figure, so the curl should be zero everywhere.

For your interest, this is the vector field $\mathbf{F}(x, y, z) = (x, y, 0)$. Its divergence is $\boldsymbol{\nabla} \cdot \mathbf{F} = 2$, which is positive everywhere, as expected. Its curl is $\boldsymbol{\nabla} \times \mathbf{F} = (0, 0, 0)$, which is zero everywhere as expected (the fuild is irrotational).

4.6 Exact and closed k-forms

We define exact and closed k-forms using the exterior derivative. We show that exact forms are always closed, and determine when closed forms are exact. We rephrase these statements in the language of vector calculus.

Objectives

You should be able to:

- Define closed and exact k-forms using the exterior derivative.
- Determine when a k-form is closed or exact, focusing on one-, two-, and three-forms in $\mathbb{R}^3$.
- Show that the general definition of closeness for one-forms reduces to our previous definition in terms of partial derivatives.
- Show that exact k-forms are always closed.
- Recall the statement of Poincare's lemma.
- Rephrase and use these statements in the language of vector calculus.

4.6.1 Exact and closed k-forms

We introduced the notion of exact one-forms in Definition 2.2.5, using the concept of differential. We also introduced closed one-forms in $\mathbb{R}^2$ in Definition 2.2.9 and in $\mathbb{R}^3$ in Definition 2.2.14, but our definition was rather ad hoc. These concepts are much more natural now that we have introduced the exterior derivative.

Definition 4.6.1 Exact and closed k-forms. Let ω be a k-form on $U \subseteq \mathbb{R}^n$. We say that ω is **closed** if $d\omega = 0$. We say that it is **exact** if there exists a $(k-1)$-form η on U such that $\omega = d\eta$. ♢

Example 4.6.2 Exact and closed one-forms in $\mathbb{R}^3$. Before we move on, let us show that this reproduces the definitions that we used in Definition 2.2.5 and Definition 2.2.14 for one-forms in $\mathbb{R}^3$.

First, if ω is a one-form, according to Definition 4.6.1 is it exact if there exists a zero-form f such that $\omega = df$. As we know that the exterior derivative of a zero-form is the same thing as the differential of a function introduced in Definition 2.2.1, this is precisely the definition of an exact one-form that we gave in Definition 2.2.5.

As for closeness, according to Definition 4.6.1 a one-form ω is closed if $d\omega = 0$. If ω is a one-form on $U \subset \mathbb{R}^3$, we can write $\omega = f\ dx + g\ dy + h\ dz$. Then

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\ dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\ dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\ dx \wedge dy.$$

Thus $d\omega = 0$ if and only if

$$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial z}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \qquad \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y},$$

which is precisely the condition stated in Definition 2.2.14. In fact, this explains where this strange condition comes from! □

As for one-forms, it is easy to show that exact forms are always closed: it follows directly from the fact that $d^2 = 0$, which is a key property of the exterior derivative proved in Lemma 4.3.9.

Lemma 4.6.3 Exact k-forms are closed. *If a k-form ω on $U \subseteq \mathbb{R}^n$ is exact, then it is closed.*

Proof. This is a direct consequence of the fact that $d^2 = 0$. If ω is exact, then there exists a $(k-1)$-form η such that $\omega = d\eta$. But then

$$d\omega = d(d\eta) = 0$$

by Lemma 4.3.9. Therefore ω is closed. ■

As for one-forms, the converse statement is much more subtle. When are closed k-forms exact? The answer depends on the domain of definition of the k-form. However, there is a simple case for which closed k-forms are always exact, as in Theorem 3.6.1. This is called "Poincare's lemma" for k-forms. We will state the result here without proof.

Theorem 4.6.4 Poincare's lemma for k-forms, version 1. *Let ω be a k-form defined on all of $\mathbb{R}^n$. Then ω is exact if and only if it is closed.*

In fact, the theorem can be generalized slightly, as in Theorem 3.6.4. What matters in the proof is not so much that ω is defined on all of $\mathbb{R}^n$, but rather that it is defined on a domain U that is contractible, which intuitively means that it can be continuously shrunk to a point within U. A more precise statement of Poincare's lemma could then be formulated as follows.

Theorem 4.6.5 Poincare's lemma for k-forms, version II. *Let ω be a k-form defined on an open ball $U \subseteq \mathbb{R}^n$. Then ω is exact if and only if it is closed.*

It is important to note however that if U is not an open ball or the whole of $\mathbb{R}^n$, then closed forms may not necessarily be exact.

Remark 4.6.6 We should note that contrary to Poincare's lemma for one-forms in Theorem 3.6.4, for $k \geq 2$ it is not true that closed k-forms on simply connected open subsets $U \subseteq \mathbb{R}^n$ are necessarily exact. This is only true for one-forms; simple-connectedness is not sufficient for $k \geq 2$. What we need is a higher-dimensional analog of simple-connectedness; this is why it was replaced by the statement that U is an open ball in Theorem 4.6.5. (Slightly more generally, one could say that U must be "contractible". Every contractible space is simply connected, but not the other way around. This is the kind of thing that is studied in cohomology and homology, see Remark 4.6.7.)

Just to highlight the subtlety here, consider the two-form

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy),$$

which is defined on $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$, which is simply connected. One can check that ω is closed, but not exact. Does that contradict Poincare's lemma? Fortunately it doesn't, as U is not an open ball in $\mathbb{R}^3$ (as it does not contain the origin). But it shows that there exists k-forms with $k \geq 2$ defined on simply connected open sets that are closed but not exact.

Remark 4.6.7 The world of cohomology (this is just for fun and beyond the scope of this class!). In fact, the relation between closed and exact forms is quite deep. As we have seen, it is closely connected to the existence of "holes" in a space, which is the subject of topology. In fact, studying when closed forms are not exact gives rise to the topic of cohomology, which is an important branch of geometry and topology. Believe me, cohomology is all over the place. You wouldn't believe it, but it is even used to describe gauge theories in physics!

While describing cohomology is way beyond the scope of this course, let me explain in a few words what it is about, just for fun, in the context of differential forms. Suppose that ω and η are closed k-forms on U. If they differ from each other by an exact form, that is $\omega - \eta = d\rho$ for some $(k-1)$-form ρ, then we say that ω and η are "equivalent" (or "cohomologous"). In this way, we construct equivalence classes of closed k-forms that differ by an exact form. Those equivalence classes (which are called "cohomology classes") form a vector space, which is called the "de Rham cohomology space" $H^k(U)$.[1]

If U is the whole of $\mathbb{R}^n$, or an open ball in $\mathbb{R}^n$, then all closed k-forms are exact, and hence they are all equivalent to the zero k-form. The de Rham cohomology spaces $H^k(U)$ (with $k \geq 1$) are then all trivial (the zero vector space). So the de Rham cohomology spaces $H^k(U)$ with $k \geq 1$ control, in a sense, how topologically non-trivial U is.

For instance, if we consider $U = \mathbb{R}^2 \setminus \{(0,0)\}$, one can show that

$$H^1(U) = \mathbb{R}, \qquad H^2(U) = 0.$$

The fact that $H^2(U) = 0$ says that all (closed) two-forms on U are exact. However, the interesting fact here is that $H^1(U) = \mathbb{R}$, which says that not all closed one-forms are exact; indeed, we saw one example of such a one-form in Example 2.2.13, which was closed but not exact. Since $H^1(U)$ is one-dimensional, this means that all closed one-forms that are not exact differ from the one we saw in that example by an exact form (they are in the same cohomology class), up to overall rescaling.

4.6.2 Translation into vector calculus for $\mathbb{R}^3$

Focusing on $\mathbb{R}^3$, we can rephrase the notions of exactness and closeness in terms of vector calculus objects, using Table 4.1.11. In $\mathbb{R}^3$, we focus on zero-, one-, two-, and three-forms. In fact, the interesting statements are for one-forms and two-forms.

We already translated the exactness and closeness statements for one-forms in the language of vector calculus in Section 2.2, but let summarize the statements here.

1. A one-form ω on $\mathbb{R}^3$ is **exact** if and only if its associated vector field $\mathbf{F}$ is **conservative**, which is the statement that
$$\mathbf{F} = \boldsymbol{\nabla} f$$
for some potential function f.

[1] Mathematically, what we are doing here is constructing a quotient vector space. If we write $Z^k(U)$ for the vector space of closed k-forms on U, and $B^k(U)$ for the vector space of exact k-forms on U, then the de Rham cohomology space is constructed as the quotient space $H^k(U) = Z^k(U)/B^k(U)$.

2. A one-form ω on $\mathbb{R}^3$ is **closed** if and only if its associated vector field $\mathbf{F}$ is **curl-free**, that is
$$\boldsymbol{\nabla} \times \mathbf{F} = 0$$
(see Remark 4.4.5).

3. The fact that exact one-forms are closed is the statement that *if* $\mathbf{F}$ *is conservative, then it is curl-free*, namely $\boldsymbol{\nabla} \times \mathbf{F} = 0$. We call this the "screening test for conservative vector fields". However, while conservative vector fields are curl-free, curl-free vector fields are not necessarily conservative.

4. Poincare's lemma translates into the statement that if $\mathbf{F}$ is defined and has continuous first order partial derivatives on all of $\mathbb{R}^3$ (or an open simply connected subset therein), then $\mathbf{F}$ *is conservative if and only if it is curl-free.*

We can do a similar translation for two-forms. The result is the following four statements.

1. A two-form ω on $\mathbb{R}^3$ is **exact** if and only if its associated vector field $\mathbf{F}$ **has a vector potential**, which is a vector field $\mathbf{G}$ such that
$$\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}.$$

2. A two-form ω on $\mathbb{R}^3$ is **closed** if and only if its associated vector field $\mathbf{F}$ is **divergence-free**, that is
$$\boldsymbol{\nabla} \cdot \mathbf{F} = 0.$$

3. The fact that exact two-forms are closed is the statement that *if there exists a vector potential for* $\mathbf{F}$*, then* $\mathbf{F}$ *is divergence-free*, namely $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. We call this the "screening test for vector potentials". However, while vector fields with a vector potential are divergence-free, divergence-free vector fields do not necessarily have a vector potential.

4. Poincare's lemma translates into the statement that if $\mathbf{F}$ is defined and has continuous first order partial derivatives on all of $\mathbb{R}^3$ (or an open ball therein), then $\mathbf{F}$ *has a vector potential if and only if it is divergence-free.*

Remark 4.6.8 Suppose that a vector field $\mathbf{F}$ on $\mathbb{R}^3$ has a vector potential $\mathbf{G}$. This means that there exists a vector field $\mathbf{G}$ such that
$$\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}.$$
Finding $\mathbf{G}$ is not always obvious, as one would need in principle to integrate fairly complicated partial differential equations. Moreover, $\mathbf{G}$ is far from unique.

It turns out that there is a nice result that drastically simplifies calculations. One can show that, if $\mathbf{F}$ has a vector potential $\mathbf{G}$, then we can always choose $\mathbf{G}$ to have vanishing z-component function. In other words, if $\mathbf{F}$ has a vector potential, then there always exists a $\mathbf{G} = (g_1, g_2, 0)$ such that
$$\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}.$$
This is very helpful when trying to find a vector potential.

In the language of differential forms, this corresponds to the statement that if ω is an exact two-form on $\mathbb{R}^3$, then there exists a one-form $\eta = f\ dx + g\ dy$ (with vanishing third component function) such that $\omega = d\eta$. We prove this statement in Exercise 4.6.3.6.

4.6.3 Exercises

1. Let ω be the following two-form on $\mathbb{R}^3$:

$$\omega = (e^x + y)\ dy \wedge dz + (ye^x + e^z)\ dz \wedge dx + (e^{x+y} - 2ze^x)\ dx \wedge dy.$$

Determine whether ω is exact. If it is, find a one-form η such that $d\eta = \omega$.

Solution. First, we notice that ω is defined on all of $\mathbb{R}^3$, so Poincare's lemma applies. So we know that it is exact if and only if it is closed. Let us first determine whether it is closed.

We calculate:

$$\begin{aligned} d\omega =& \frac{\partial}{\partial x}(e^x + y)\ dx \wedge dy \wedge dz + \frac{\partial}{\partial y}(ye^x + e^z)\ dy \wedge dz \wedge dx \\ &+ \frac{\partial}{\partial z}(e^{x+y} - 2ze^x)\ dz \wedge dx \wedge dy \\ =&(e^x + e^x - 2e^x)\ dx \wedge dy \wedge dz \\ =&0. \end{aligned}$$

Thus ω is closed, and hence, by Poincare's lemma, it is also exact.

We now want to find a one-form η such that $d\eta = \omega$. We assume that η takes the form

$$\eta = f\ dx + g\ dy.$$

Its exterior derivative is

$$d\eta = -\frac{\partial g}{\partial z}\ dy \wedge dz + \frac{\partial f}{\partial z}\ dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\ dx \wedge dy.$$

So we need to solve the three equations:

$$-\frac{\partial g}{\partial z} = e^x + y, \qquad \frac{\partial f}{\partial z} = ye^x + e^z, \qquad \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = e^{x+y} - 2ze^x.$$

Integrating the first one, we get:

$$g(x, y, z) = -ze^x - yz + \alpha(x, y)$$

for some function $\alpha(x, y)$. Integrating the second one, we get

$$f(x, y, z) = yze^x + e^z + \beta(x, y)$$

for some function $\beta(x, y)$. Substituting these expressions in the third equation, we get:

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -ze^x + \frac{\partial \alpha(x, y)}{\partial x} - ze^x - \frac{\partial \beta(x, y)}{\partial y} = e^{x+y} - 2ze^x,$$

so we must have

$$\frac{\partial \alpha(x, y)}{\partial x} - \frac{\partial \beta(x, y)}{\partial y} = e^{x+y}.$$

We can choose any function α and β that work. So why not choose $\beta = 0$, and $\alpha(x, y) = e^{x+y}$. Then we get that the one-form

$$\eta = (yze^x + e^z)\ dx + (-ze^x - yz + e^{x+y})\ dy$$

is such that $d\eta = \omega$.

2. Let ω be the following two-form on $\mathbb{R}^3$:

$$\omega = xy\ dy \wedge dz + yz\ dz \wedge dx + zx\ dx \wedge dy.$$

Determine whether ω is exact. If it is, find a one-form η such that $d\eta = \omega$.

Solution. ω is defined on all of $\mathbb{R}^3$, so Poincare's lemma applies. Thus ω is exact if and only if it is closed. Let us determine whether it is closed.

We calculate:

$$\begin{aligned} d\omega =& \frac{\partial}{\partial x}(xy)\ dx \wedge dy \wedge dz + \frac{\partial}{\partial y}(yz)\ dy \wedge dz \wedge dx + \frac{\partial}{\partial z}(zx)\ dz \wedge dx \wedge dy \\ =&(y + z + x)\ dx \wedge dy \wedge dz. \end{aligned}$$

As this is non-zero, ω is not closed. We then conclude that ω is not exact.

3. Let $\mathbf{F}$ be the following vector field on $\mathbb{R}^3$:

$$\mathbf{F} = (yz, xz, xy).$$

Determine whether $\mathbf{F}$ has a vector potential $\mathbf{G}$. If it does, find such a vector potential.

Solution. Since the component functions of $\mathbf{F}$ are smooth on all of $\mathbb{R}^3$, Poincare's lemma applies. So $\mathbf{F}$ will have a vector potential if and only if it is divergence-free. We calculate its divergence:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} =& \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) \\ =&0, \end{aligned}$$

and thus we conclude that there exists a vector potential $\mathbf{G}$ such that $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{F}$.

Now we need to find $\mathbf{G}$. We assume that $\mathbf{G} = (g_1, g_2, 0)$. Then the condition that $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{F}$ is

$$\boldsymbol{\nabla} \times \mathbf{G} = \left(-\frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) = (yz, xz, xy).$$

So we get three equations to solve. We integrate the first one (equality of the x-component functions) to get:

$$g_2(x, y, z) = -\frac{yz^2}{2} + \alpha(x, y).$$

We integrate the second one (equality of the y-component functions) to get:

$$g_1(x, y, z) = \frac{xz^2}{2} + \beta(x, y).$$

Substituting in the third one (equality of the z-component functions), we get:

$$\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} = \frac{\partial \alpha(x, y)}{\partial x} - \frac{\partial \beta(x, y)}{\partial y} = xy.$$

We need to find any two functions α and β that satisfy this condition. We choose $\beta = 0$, $\alpha = \frac{x^2 y}{2}$. As a result, we have found a vector potential

$$\mathbf{G}(x, y, z) = \left(\frac{xz^2}{2}, \frac{y}{2}(x^2 - z^2), 0\right).$$

4. Consider the two-form

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy),$$

which is defined on $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$. Show that ω is closed.

Solution. We want to show that $d\omega = 0$. First, we calculate

$$\begin{aligned} \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} =& \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\ =& \frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}}. \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{(x^2+y^2+z^2) - 3y^2}{(x^2+y^2+z^2)^{5/2}}$$

and

$$\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{(x^2+y^2+z^2) - 3z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Now, we get:

$$\begin{aligned} d\omega =& \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}}\ dx \wedge dy \wedge dz + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}}\ dy \wedge dz \wedge dx \\ &+ \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}\ dz \wedge dx \wedge dy \\ =& \frac{3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{5/2}}\ dx \wedge dy \wedge dz \\ =& 0. \end{aligned}$$

We conclude that ω is closed.

What is interesting with this example is that ω is closed, but one can show that it is not exact. This doesn't contradict Poincare's lemma, as ω is not defined on all of $\mathbb{R}^3$ (or an open ball thereof). But it is interesting since ω is defined on a simply connected subset of $\mathbb{R}^3$, so it shows that there exists k-forms with $k \geq 2$ defined on simply connected subsets that are closed but not exact.

5. Consider the vector field

$$\mathbf{F}(x,y,z) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\right).$$

(a) Find the domain of definition of $\mathbf{F}$. Is it path connected? Simply connected?

(b) Determine the divergence of $\mathbf{F}$.

(c) Determine the curl of $\mathbf{F}$.

(d) Does $\mathbf{F}$ have a vector potential? Justify your answer. If it does, find such a vector potential.

(e) Is $\mathbf{F}$ conservative? Justify your answer. If it is, find a potential function.

Solution.

(a) $\mathbf{F}$ is defined (and in fact, is smooth) wherever the denominator is non-zero. This is wherever $(x, y) \neq (0, 0)$. So the domain of definition of $\mathbb{F}$ is

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}.$$

This is $\mathbb{R}^3$ minus the z-axis. It is path connected, since any two points in U can be connected by a path. It is however not simply connected, since a closed curve around the z-axis cannot be continuously contracted to a point within U (it would hit the z-axis, which is not in U).

(b) We calculate the divergence of $\mathbf{F}$:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{F} =& \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial z}(z) \\
=& \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} + 1 \\
=& 1.
\end{aligned}$$

(c) We calculate the curl of $\mathbf{F}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} =& \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right), \frac{\partial}{\partial z}\left(-\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(z),\right. \\
& \left.\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right) \\
=& \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right) \\
=& (0, 0, 0).
\end{aligned}$$

So $\mathbf{F}$ is curl-free.

(d) We found in (b) that the divergence of $\mathbf{F}$ is non-zero. This means that $\mathbf{F}$ cannot have a vector potential, since vector fields that have a vector potential are divergence-free.

(e) We found in (c) that $\mathbf{F}$ is curl-free, so it passes the screening test for conservative vector fields. However, since its domain of definition U is not simply connected, Poincare's lemma does not apply. We thus cannot conclude whether $\mathbf{F}$ is conservative from the statement that it is curl-free.

In fact, we can show that it is not conservative by showing that its integral along a closed loop is non-zero, as in Exercise 3.4.3.2. Let ω be the one-form associated to $\mathbf{F}$:

$$\omega = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy + z\, dz.$$

Consider the parametric curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ with $\alpha(t) = (\cos(t), \sin(t), 0)$, which is the unit circle (counterclockwise) around the origin in the xy-plane. The pullback of ω is:

$$\begin{aligned}\alpha^*\omega =& \left(-\frac{\sin(t)}{\cos^2(t) + \sin^2(t)}(-\sin(t)) + \frac{\cos(t)}{\cos^2(t) + \sin^2(t)}\cos(t)\right)\ dt\\ =& dt.\end{aligned}$$

The line integral of ω along α is thus:

$$\int_\alpha \omega = \int_0^{2\pi} \alpha^*\omega = \int_0^{2\pi} dt = 2\pi.$$

Since this is non-zero, by Corollary 3.4.3 (or, in other words, the Fundamental Theorem for line integrals), ω cannot be exact, since the line integrals of exact one-forms along closed curves vanish. Equivalently, the vector field $\mathbf{F}$ is not conservative.

6. Let ω be an exact two-form on $\mathbb{R}^3$. Show that there exists a one-form η of the form

$$\eta = f\ dx + g\ dy$$

(i.e. with a vanishing z-component function) such that $d\eta = \omega$.

In other words, in the language of vector calculus, if a vector field $\mathbf{F}$ has a vector potential $\mathbf{G}$, then $\mathbf{G}$ can always be chosen to take the form $\mathbf{G} = (g_1, g_2, 0)$.

Solution. Since ω is exact, we know that there exists a one-form β such that $d\beta = \omega$. Suppose that we find such a β:

$$\beta = b_1\ dx + b_2\ dy + b_3\ dz,$$

for some component functions b_1, b_2, b_3. β is certainly not unique; there are many one-forms such that their exterior derivative equals ω. In fact, since $d^2 = 0$, we can add to β any exact one-form dF for a function F, and we get another one-form whose exterior derivative is ω. That is, if we define

$$\tilde{\beta} = \beta - dF$$

for any function F, then $d\tilde{\beta} = d\beta - d^2F = \omega$.

In particular, since

$$dF = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy + \frac{\partial F}{\partial z}\ dz,$$

if we can choose F such that

$$\frac{\partial F}{\partial z} = b_3,$$

then we see that

$$\tilde{\beta} = \left(b_1 - \frac{\partial F}{\partial x}\right)\ dx + \left(b_2 - \frac{\partial F}{\partial y}\right)\ dy,$$

and hence it is of the form that we are looking for (no z-component function). But this is easy to do; simply pick

$$F(x, y, z) = \int b_3(x, y, z)\ dz,$$

i.e. any antiderivative in the z-variable will do. So we conclude that we can always find a one-form $\tilde{\beta}$ with no z-component and such that $d\tilde{\beta} = \omega$.

4.7 The pullback of a k-form

In Section 2.4 we defined the pullback of a one-form. We now generalize this concept to k-forms. We first do it using an axiomatic approach as for one-forms, and relate the result to the concept of Jacobian. We also introduce a more direct definition of pullback using the algebraic approach to basic k-forms introduced in Subsection 4.1.1, and show that it is consistent with our axiomatic approach..

Objectives

You should be able to:

- Determine the pullback of a k-form.
- Show that the pullback of a 2-form in $\mathbb{R}^2$ and a 3-form in $\mathbb{R}^3$ is given by the Jacobian determinant.

4.7.1 The pullback of a k-form: an axiomatic approach

In Section 2.4 we defined the pullback of a one-form. We used an "axiomatic" approach: we first defined the properties that we wanted the pullback to satisfy, and showed that there is a unique construction that satisfies these properties. The properties that we specified were, in words:

1. The pullback of a sum of one-forms is the sum of the pullbacks.
2. The pullback of a product of a zero-form and a one-form is the product of the pullbacks.
3. The pullback of the exterior derivative of a zero-form is the exterior derivative of the pullback.

In this section we define the pullback of a k-form using a similar approach. All we need to do is generalize the first and second properties to k-forms.

More precisely, let $\phi : V \to U$ be a smooth function, with $V \subseteq \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ open subsets. We want the pullback ϕ^* to satisfy the following properties:

1. If ω and η are k-forms on U, then
$$\phi^*(\omega + \eta) = \phi^*\omega + \phi^*\eta.$$
2. If ω is a k-form and η an ℓ-form on U, then
$$\phi^*(\omega \wedge \eta) = (\phi^*\omega) \wedge (\phi^*\eta).$$
3. If f is a zero-form (a function) on U, then
$$\phi^*(df) = d(\phi^* f).$$

The third property is unchanged, and the first two are naturally generalized to k-forms.

Imposing these properties gives us a unique definition for the pullback of a k-form.

Lemma 4.7.1 The pullback of a k-form. *Let $\phi : V \to U$ be a smooth function, with $V \subseteq \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ open subsets. We write $\mathbf{t} = (t_1, \ldots, t_m) \in V$, and $\phi(\mathbf{t}) = (x_1(\mathbf{t}), \ldots, x_n(\mathbf{t}))$. Let ω be a k-form on U, with*

$$\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \cdots i_k} \; dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

*for some functions $f_{i_1 \cdots i_k} : U \to \mathbb{R}$. Then the **pullback** $\phi^*\omega$ is a k-form on V given by:*

$$\phi^*\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \cdots i_k}(\phi(\mathbf{t})) \left(\frac{\partial x_{i_1}}{\partial t_1} \; dt_1 + \ldots + \frac{\partial x_{i_1}}{\partial t_m} \; dt_m \right) \wedge \cdots \wedge \left(\frac{\partial x_{i_k}}{\partial t_1} \; dt_1 + \ldots + \frac{\partial x_{i_k}}{\partial t_m} \; dt_m \right).$$

Proof. Start with a basic k-form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since we impose (Property 2) that $\phi^*(\omega \wedge \eta) = (\phi^*\omega) \wedge \phi^*\eta)$, we know that

$$\phi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \phi^*(dx_{i_1}) \wedge \cdots \wedge \phi^*(dx_{i_k})..$$

But we already calculated how basic one-forms transform in Lemma 2.4.4; they transform as differentials would. This calculation followed from Property 3, which we still impose here, so it is still valid. This gives us the pullback of basic k-forms.

Then we use Property 2 to extended it to a function times a basic k-form, and then Property 1 to extend it to linear combination of such terms, to get the final result for the pullback of a generic k-form. ■

This formula certainly looks ugly, but it is really not that bad. Concretely, all you need to do is compose the component functions with ϕ, and transform the basic one-forms one by one in the wedge product as in Lemma 2.4.4, i.e. they transform as you would expect differentials transform. That's it! This will certainly be clearer with examples.

Example 4.7.2 The pullback of a two-form. Consider the following two-form on $\mathbb{R}^3$:

$$\omega = xy \; dy \wedge dz + (xz + y) \; dz \wedge dx + dx \wedge dy,$$

and the smooth function $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\phi(u, v) = (uv, u^2, v^2) = (x(u, v), y(u, v), z(u, v)).$$

To calculate $\phi^*\omega$, let us start by calculating the pullback of the basic one-forms. We get:

$$\begin{aligned} \phi^*(dx) =& \frac{\partial x}{\partial u} \; du + \frac{\partial x}{\partial v} \; dv = v \; du + u \; dv, \\ \phi^*(dy) =& \frac{\partial y}{\partial u} \; du + \frac{\partial y}{\partial v} \; dv = 2u \; du, \\ \phi^*(dz) =& \frac{\partial z}{\partial u} \; du + \frac{\partial z}{\partial v} \; dv = 2v \; dv. \end{aligned}$$

Putting this together, we get:

$$\begin{aligned}\phi^*\omega =&(uv)(u^2)\phi^*(dy)\wedge\phi^*(dz)+((uv)(v^2)+u^2)\phi^*(dz)\wedge\phi^*(dx)+\phi^*(dx)\wedge\phi^*(dy)\\ =&u^3v(2u\ du)\wedge(2v\ dv)+(uv^3+u^2)(2v\ dv)\wedge(v\ du+u\ dv)+(v\ du+u\ dv)\wedge(2u\ du)\\ =&4u^4v^2\ du\wedge dv+2uv^2(v^3+u)\ dv\wedge du+2u^2\ dv\wedge du\\ =&(4u^4v^2-2uv^5-2u^2v^2-2u^2)du\wedge dv.\end{aligned}$$

Note that we used the fact that $du\wedge du = dv\wedge dv = 0$, and $dv\wedge du = -du\wedge dv$. □

Example 4.7.3 The pullback of a three-form. Consider the following three-form on $\mathbb{R}^3$:

$$\omega = e^{x+y+z}dx\wedge dy\wedge dz,$$

and the smooth function $\phi:\mathbb{R}^3_{>0}\to\mathbb{R}^3$ given by

$$\phi(u,v,w)=(\ln(uv),\ln(vw),\ln(wu)).$$

The pullback of the basic one-forms is:

$$\begin{aligned}\phi^*(dx)=&\frac{1}{u}\ du+\frac{1}{v}\ dv,\\ \phi^*(dy)=&\frac{1}{v}\ dv+\frac{1}{w}\ dw,\\ \phi^*(dz)=&\frac{1}{w}\ dw+\frac{1}{u}\ du.\end{aligned}$$

We get:

$$\begin{aligned}\phi^*\omega =&e^{\ln(uv)+\ln(vw)+\ln(wu)}\left(\frac{1}{u}\ du+\frac{1}{v}\ dv\right)\wedge\left(\frac{1}{v}\ dv+\frac{1}{w}\ dw\right)\wedge\left(\frac{1}{w}\ dw+\frac{1}{u}\ du\right)\\ =&u^2v^2w^2\left(\frac{1}{uvw}\ du\wedge dv\wedge dw+\frac{1}{uvw}\ dv\wedge dw\wedge du\right)\\ =&2uvw\ du\wedge dv\wedge dw,\end{aligned}$$

where we used the fact that the basic three-forms vanish whenever one of the factor is repeated, and $dv\wedge dw\wedge du = du\wedge dv\wedge dw$ since we need to exchange two basic one-forms twice to relate the two basic three-forms. □

Good, so we are now experts at computing pullbacks! Calculating the pullback of a k-form is not more difficult than calculating the pullback of a one-form, but the calculation may be longer, and you need to use the anti-symmetry properties of basic k-forms in Lemma 4.1.6 to simplify the result at the end of your calculation.

Property 3 in our axiomatic definition states that the pullback commutes with the exterior derivative for zero-form. It turns out that this property, which is very important, holds in general for k-forms. Let us prove that.

Lemma 4.7.4 The pullback commutes with the exterior derivative. *Let $\phi: V\to U$ be a smooth function, with $V\subseteq\mathbb{R}^m$ and $U\subseteq\mathbb{R}^n$ open subsets. Let ω be a k-form on U. Then:*

$$\phi^*(d\omega)=d(\phi^*\omega).$$

Proof. Recall the definition of the exterior derivative Definition 4.3.1. Let $\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then

$$d\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} d(f_{i_1\cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Using Properties 1 and 2, we can write

$$\phi^*(d\omega) = \sum_{1\leq i_1<\cdots<i_k\leq n} \phi^*(d(f_{i_1\cdots i_k})) \wedge \phi^*(dx_{i_1}) \wedge \cdots \wedge \phi^*(dx_{i_k}).$$

Now we can use Property 3, which states that

$$\phi^*(d(f_{i_1\cdots i_k})) = d(\phi^* f_{i_1\cdots i_k}),$$

since the $f_{i_1\cdots i_k}$ are just functions (zero-forms). Thus we have:

$$\begin{aligned}\phi^*(d\omega) =& \sum_{1\leq i_1<\cdots<i_k\leq n} d(\phi^* f_{i_1\cdots i_k}) \wedge \phi^*(dx_{i_1}) \wedge \cdots \wedge \phi^*(dx_{i_k})\\ =&d\left(\sum_{1\leq i_1<\cdots<i_k\leq n} (\phi^* f_{i_1\cdots i_k})\phi^*(dx_{i_1}) \wedge \cdots \wedge \phi^*(dx_{i_k})\right)\\ =&d(\phi^*\omega),\end{aligned}$$

where the last line follows from Properties 1 and 2 again. ■

4.7.2 The pullback of a top form in $\mathbb{R}^n$ and the Jacobian determinant

There is a special case for which the pullback takes a very nice form. This case will play a role shortly in our theory of integration, as it will be related to the transformation formula (the generalization of the substitution formula) for multiple integrals.

Let us start by recalling the definition of the Jacobian of a smooth function.

Definition 4.7.5 The Jacobian. Let $\phi : V \to U$ be a smooth function with $U, V \subseteq \mathbb{R}^n$ open subsets. Let us write $\mathbf{t} = (t_1, \ldots, t_n) \in V$, and

$$\phi(\mathbf{t}) = (x_1(\mathbf{t}), \ldots, x_n(\mathbf{t})).$$

The **Jacobian** of ϕ, which we denote by $\frac{\partial(x_1,\ldots,x_n)}{\partial(t_1,\ldots,t_n)}$ or J_ϕ or $D\phi$ (lots of different notations!), is the $n \times n$ matrix of first partial derivatives:

$$D\phi = J_\phi = \frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_n)} = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_n} \end{pmatrix}$$

Its determinant is called the **Jacobian determinant**. ◊

It turns out that if we pullback an n-form with respect to such a ϕ, we can write $\phi^*\omega$ in terms of the Jacobian determinant. First, let us introduce the common name “top form” for n-form on open subsets $U \subseteq \mathbb{R}^n$.

Definition 4.7.6 Top form. We call an n-form on an open subset $U \subseteq \mathbb{R}^n$ a **top form**. $\Diamond$

Such forms are called "top forms" because all forms with $k \geq n$ necessarily vanish on $\mathbb{R}^n$. Going back to the pullback, we get the nice following result when we pullback a top form:

Lemma 4.7.7 The pullback of a top form in $\mathbb{R}^n$ in terms of the Jacobian determinant. *Suppose that ω is a top form on $U \subseteq \mathbb{R}^n$, and $\phi : V \to U$ a smooth function with $V \subseteq \mathbb{R}^n$. As above we write*

$$\phi(\mathbf{t}) = (x_1(\mathbf{t}), \ldots, x_n(\mathbf{t}))$$

with $\mathbf{t} = (t_1, \ldots, t_n) \in V$, and

$$\omega = f \ dx_1 \wedge \ldots \wedge dx_n$$

for some function $f : U \to \mathbb{R}$. Then

$$\phi^* \omega = f(\phi(\mathbf{t}))(\det J_\phi) \ dt_1 \wedge \ldots \wedge dt_n,$$

where $\det J_\phi$ is the Jacobian determinant.

Proof for $\mathbb{R}^2$ and $\mathbb{R}^3$. It is not so easy to write a general proof for $\mathbb{R}^n$ using the computational approach for the pullback that we have used so far. We will be able to write down a general proof easily in the next subsection after having introduced a more algebraic approach to the pullback. For the time being, let us prove the statement explicitly for $\mathbb{R}^2$ and $\mathbb{R}^3$.

For $\mathbb{R}^2$, our function ϕ takes the form

$$\phi(t_1, t_2) = (x_1(t_1, t_2), x_2(t_1, t_2)).$$

What we need to show is that

$$\phi^*(dx_1 \wedge dx_2) = (\det J_\phi) dt_1 \wedge dt_2,$$

where

$$\det J_\phi = \det \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \end{pmatrix} = \frac{\partial x_1}{\partial t_1} \frac{\partial x_2}{\partial t_2} - \frac{\partial x_2}{\partial t_1} \frac{\partial x_1}{\partial t_2}.$$

But

$$\begin{aligned} \phi^*(dx_1 \wedge dx_2) &= \left(\frac{\partial x_1}{\partial t_1} \ dt_1 + \frac{\partial x_1}{\partial t_2} \ dt_2 \right) \wedge \left(\frac{\partial x_2}{\partial t_1} \ dt_1 + \frac{\partial x_2}{\partial t_2} \ dt_2 \right) \\ &= \left(\frac{\partial x_1}{\partial t_1} \frac{\partial x_2}{\partial t_2} - \frac{\partial x_2}{\partial t_1} \frac{\partial x_1}{\partial t_2} \right) dt_1 \wedge dt_2, \end{aligned}$$

and the lemma is proved.

The calculation is similar but more involved for $\mathbb{R}^3$. We have:

$$\phi(t_1, t_2, t_3) = (x_1(t_1, t_2, t_3), x_2(t_1, t_2, t_3), x_3(t_1, t_2, t_3)).$$

What we need to show is that

$$\phi^*(dx_1 \wedge dx_2 \wedge dx_3) = (\det J_\phi) dt_1 \wedge dt_2 \wedge dt_3,$$

where

$$\begin{aligned}\det J_\phi =&\det\begin{pmatrix}\frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \frac{\partial x_1}{\partial t_3}\\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_2}{\partial t_3}\\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3}\end{pmatrix}\\ =&\frac{\partial x_1}{\partial t_1}\frac{\partial x_2}{\partial t_2}\frac{\partial x_3}{\partial t_3}+\frac{\partial x_1}{\partial t_2}\frac{\partial x_2}{\partial t_3}\frac{\partial x_3}{\partial t_1}+\frac{\partial x_1}{\partial t_3}\frac{\partial x_2}{\partial t_1}\frac{\partial x_3}{\partial t_2}\\ &-\frac{\partial x_1}{\partial t_2}\frac{\partial x_2}{\partial t_1}\frac{\partial x_3}{\partial t_3}-\frac{\partial x_1}{\partial t_1}\frac{\partial x_2}{\partial t_3}\frac{\partial x_3}{\partial t_2}-\frac{\partial x_1}{\partial t_3}\frac{\partial x_2}{\partial t_2}\frac{\partial x_3}{\partial t_1}.\end{aligned}$$

But

$$\begin{aligned}\phi^*(dx_1\wedge dx_2\wedge dx_3)=&\left(\frac{\partial x_1}{\partial t_1}\,dt_1+\frac{\partial x_1}{\partial t_2}\,dt_2+\frac{\partial x_1}{\partial t_3}\,dt_3\right)\\ &\wedge\left(\frac{\partial x_2}{\partial t_1}\,dt_1+\frac{\partial x_2}{\partial t_2}\,dt_2+\frac{\partial x_2}{\partial t_3}\,dt_3\right)\\ &\wedge\left(\frac{\partial x_3}{\partial t_1}\,dt_1+\frac{\partial x_3}{\partial t_2}\,dt_2+\frac{\partial x_3}{\partial t_3}\,dt_3\right)\\ =&\Big(\frac{\partial x_1}{\partial t_1}\frac{\partial x_2}{\partial t_2}\frac{\partial x_3}{\partial t_3}+\frac{\partial x_1}{\partial t_2}\frac{\partial x_2}{\partial t_3}\frac{\partial x_3}{\partial t_1}+\frac{\partial x_1}{\partial t_3}\frac{\partial x_2}{\partial t_1}\frac{\partial x_3}{\partial t_2}\\ &-\frac{\partial x_1}{\partial t_2}\frac{\partial x_2}{\partial t_1}\frac{\partial x_3}{\partial t_3}-\frac{\partial x_1}{\partial t_1}\frac{\partial x_2}{\partial t_3}\frac{\partial x_3}{\partial t_2}-\frac{\partial x_1}{\partial t_3}\frac{\partial x_2}{\partial t_2}\frac{\partial x_3}{\partial t_1}\Big)dt_1\wedge dt_2\wedge dt_3,\end{aligned}$$

which completes the proof in $\mathbb{R}^3$. Phew, this was painful. ■

The appearance of the Jacobian determinant here is quite nice. It will be related to the transformation formula for multiple integrals, when we integrate a top form over a bounded region in $\mathbb{R}^n$. We note however that we obtain the Jacobian determinant here, not its absolute value (in comparison to what you may have seen in previous calculus classes); this will be related to the fact that our theory of integration is oriented.

4.7.3 The pullback of a k-form: an algebraic approach (optional)

In this section we introduce a direct definition of the pullback using the algebraic approach to basic k-forms introduced in Subsection 4.1.1. We then show that the three fundamental properties that we used to define the pullback are satisfied, thus justifying our original axiomatic approach.

Recall from Definition 4.1.1 (naturally generalized to $\mathbb{R}^n$) that a basic one-form dx_i is a linear map $dx_i : \mathbb{R}^n \to \mathbb{R}$ which acts as

$$dx_i(u_1,\ldots,u_n)=u_i,$$

i.e. it outputs the i'th component of the vector $\mathbf{u}\in\mathbb{R}^n$. As we are now thinking of the basic one-forms dx_i as linear maps, we can define their pullback by composition, as we originally did for functions in Definition 2.4.1.

We first define and study the pullback when ϕ is a linear map between vector spaces. We will generalize to the case of a smooth function afterwards.

Definition 4.7.8 The pullback of a basic one-form with respect to a linear map. Let $dx_i : \mathbb{R}^n \to \mathbb{R}$ be a basic one-form, and let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. We define the pullback $\phi^*(dx_i) : \mathbb{R}^m \to \mathbb{R}$ by composition:

$$\phi^*(dx_i) = dx_i \circ \phi : \mathbb{R}^m \to \mathbb{R}.$$

$\diamondsuit$

Let us now give an explicit formula for the pullback of a basic one-form.

Lemma 4.7.9 An explicit formula for the pullback of a basic one-form. *Let $dx_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$ be the basic one-forms on $\mathbb{R}^n$, and $dt_j : \mathbb{R}^m \to \mathbb{R}$, $j = 1, \dots, m$ be the basic one-forms on $\mathbb{R}^m$. The linear map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ can be represented by a matrix $A = (a_{ij})$. Then we can write*

$$\phi^*(dx_i) = \sum_{j=1}^m a_{ij} dt_j = a_{i1} dt_1 + \ldots + a_{im} dt_m.$$

Proof. Write $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$. Then

$$\begin{aligned}
\phi^*(dx_i)(\mathbf{v}) =& dx_i(\phi(\mathbf{v})) \\
=& dx_i(a_{11}v_1 + \ldots + a_{1m}v_m, \cdots, a_{n1}v_1 + \ldots a_{nm}v_m) \\
=& a_{i1}v_1 + \ldots + a_{im}v_m \\
=& a_{i1}dt_1(\mathbf{v}) + \ldots + a_{im}dt_m(\mathbf{v}) \\
=& (a_{i1}dt_1 + \ldots + a_{im}dt_m)(\mathbf{v}).
\end{aligned}$$

■

It is then easy to generalize the definition of pullback to the basic k-forms.

Definition 4.7.10 The pullback of a basic k-form with respect to a linear map. Let $dx_{i_1} \wedge \ldots \wedge dx_{i_k} : (\mathbb{R}^n)^k \to \mathbb{R}$ be a basic k-form, and let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Let $\mathbf{v}^1, \dots, \mathbf{v}^k \in \mathbb{R}^m$ be vectors. We define the pullback $\phi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) : (\mathbb{R}^m)^k \to \mathbb{R}$ by:

$$\phi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k})(\mathbf{v}^1, \dots, \mathbf{v}^k) = dx_{i_1} \wedge \ldots \wedge dx_{i_k}(\phi(\mathbf{v}^1), \dots, \phi(\mathbf{v}^k)).$$

$\diamondsuit$

It follows directly from the definition that the pullback commutes with the wedge product:

Lemma 4.7.11 The pullback commutes with the wedge product. *Under the setup above,*

$$\phi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \phi^*(dx_{i_1}) \wedge \ldots \wedge \phi^*(dx_{i_k}).$$

Proof. By definition,

$$\begin{aligned}
\phi^*(dx_{i_1}) \wedge \ldots \wedge \phi^*(dx_{i_k})(\mathbf{v}^1, \dots, \mathbf{v}^k) =& dx_{i_1} \wedge \ldots \wedge dx_{i_k}(\phi(\mathbf{v}^1), \dots, \phi(\mathbf{v}^k)) \\
=& \phi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k})(\mathbf{v}^1, \dots, \mathbf{v}^k).
\end{aligned}$$

■

As a corollary, we obtain an explicit formula to calculate the pullback of a basic k-form.

Corollary 4.7.12 An explicit formula for the pullback of a basic k-form. *Let $dx_{i_1} \wedge \ldots \wedge dx_{i_k} : (\mathbb{R}^n)^k \to \mathbb{R}$ be a basic k-form, and let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Let $dt_j : \mathbb{R}^m \to \mathbb{R}$, $j = 1, \ldots, m$ be the basic one-forms on $\mathbb{R}^m$. The linear map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ can be represented by a matrix $A = (a_{ij})$. Then we can write*

$$\phi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = (a_{i_1 1} dt_1 + \ldots + a_{i_1 m} dt_m) \wedge \cdots \wedge (a_{i_k 1} dt_1 + \ldots + a_{i_k m} dt_m) .$$

We can also write down an explicit formula in terms of the determinant when we pullback a basic n-form from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Lemma 4.7.13 The pullback of a basic n-form in $\mathbb{R}^n$. *Let $dx_1 \wedge \cdots \wedge dx_n : (\mathbb{R}^n)^n \to \mathbb{R}$, and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, which can be representated by an $n \times n$ matrix $A = (a_{ij})$. Then*

$$\phi^*(dx_1 \wedge \cdots \wedge dx_n) = (\det A)\ dx_1 \wedge \cdots \wedge dx_n.$$

Proof. Let $\mathbf{v}^1, \ldots, \mathbf{v}^n \in \mathbb{R}^n$. Then

$$\begin{aligned}
\phi^*(dx_1 \wedge \cdots \wedge dx_n)(\mathbf{v}^1, \ldots, \mathbf{v}^n) =& dx_1 \wedge \cdots \wedge dx_n(\phi(\mathbf{v}^1), \ldots, \phi(\mathbf{v}^n)) \\
=& \det \begin{pmatrix} A_{11} v_1^1 + \ldots + A_{1n} v_n^1 & \cdots & A_{11} v_1^n + \ldots + A_{1n} v_n^n \\ \vdots & \ddots & \vdots \\ A_{n1} v_1^1 + \ldots + A_{nn} v_n^1 & \cdots & A_{n1} v_1^n + \ldots + A_{nn} v_n^n \end{pmatrix} \\
=& \det \left(\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_n^1 & \cdots & v_n^n \end{pmatrix} \right) \\
=& (\det A) \det \begin{pmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_n^1 & \cdots & v_n^n \end{pmatrix} \\
=& (\det A) dx_1 \wedge \ldots \wedge dx_n(\mathbf{v}^1, \ldots, \mathbf{v}^n),
\end{aligned}$$

which concludes the proof. ■

This is all very nice, but so far we only looked at basic k-forms, and linear maps $\phi : \mathbb{R}^m \to \mathbb{R}^n$. How do we generalize this to general differential k-forms on $U \subseteq \mathbb{R}^n$, and to smooth functions $\phi : V \to U$ with $V \subseteq \mathbb{R}^m$? The idea is simple. For any point $\mathbf{t} \in V$, the Jacobian matrix of ϕ (i.e. the matrix of first partial derivatives), also called the "total derivative of ϕ", provides a linear map $\mathbb{R}^m \to \mathbb{R}^n$.[1]

In other words, given a smooth function $\phi : V \to U$, if we write $\phi(\mathbf{t}) = (x_1(\mathbf{t}), \ldots, x_n(\mathbf{t}))$, we can construct the pullback of k-forms exactly as above, but with the specific linear map $\mathbb{R}^m \to \mathbb{R}^n$ given by the Jacobian matrix:

$$A = (a_{ij}) = \left(\frac{\partial x_i}{\partial t_j} \right).$$

Then we see that we recover all the formulae that we obtained previously, and that our

[1]This is called the "differential" or "total derivative" of the smooth function ϕ: in fancier differential geometry, one would say that it is a linear map from the tangent space of V at $\mathbf{t}$ to the tangent space of U at $\phi(\mathbf{t})$.

three fundamental properties are satisfied! The pullback of a basic one-form becomes

$$\phi^*(dx_i) = \frac{\partial x_i}{\partial t_1}\, dt_1 + \ldots + \frac{\partial x_i}{\partial t_n}\, dt_n,$$

as before. Property 1 is obviously satisfied by definition. Lemma 4.7.11 becomes Property 2, and Corollary 4.7.12 becomes our general formula for the pullback of k-forms in Lemma 4.7.1. It is easy to check that Property 3 is satisfied. Finally, we obtain a general proof of Lemma 4.7.7 on $\mathbb{R}^n$, as this is simply Lemma 4.7.13. Neat!

4.7.4 Exercises

1. Consider the basic two-form $dx \wedge dy$ on $\mathbb{R}^2$, and the polar coordinate transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\alpha(r, \theta) = (r\cos\theta, r\sin\theta).$$

Show by explicit calculation that

$$\alpha^*(dx \wedge dy) = r\ dr \wedge d\theta = (\det J_\alpha) dr \wedge d\theta.$$

Solution. Let us start by calculating the pullback two-form. We get:

$$\begin{aligned}\alpha^*(dx \wedge dy) =&(\cos\theta\ dr - r\sin\theta\ d\theta) \wedge (\sin\theta\ dr + r\cos\theta\ d\theta)\\ =&r\cos^2\theta\ dr \wedge d\theta - r\sin^2\theta\ d\theta \wedge dr\\ =&r(\cos^2\theta + \sin^2\theta) dr \wedge d\theta\\ =&r\ dr \wedge d\theta.\end{aligned}$$

Next, we show that $\det J_\alpha = r$. By definition of the Jacobian matrix, we have:

$$\begin{aligned}\det J_\alpha =&\det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix}\\ =&\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\\ =&r\cos^2\theta + r\sin^2\theta\\ =&r.\end{aligned}$$

Therefore,

$$\alpha^*(dx \wedge dy) = r\ dr \wedge d\theta = (\det J_\alpha) dr \wedge d\theta$$

as claimed.

2. Let

$$\omega = (x^2 + y^2 + z^2)\ dx \wedge dy \wedge dz$$

on $\mathbb{R}^3$, and $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ the spherical transformation

$$\alpha(r, \theta, \phi) = (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)).$$

(a) Show by explicit calculation that

$$\alpha^*(dx \wedge dy \wedge dz) = r^2\sin(\theta)\ dr \wedge d\theta \wedge d\phi = (\det J_\alpha) dr \wedge d\theta \wedge d\phi.$$

(b) Use this to find $\alpha^*\omega$.

Solution.

(a) We calculate the pullback:

$$\begin{aligned}\alpha^*(dx \wedge dy \wedge dz) =&(\sin(\theta)\cos(\phi)dr + r\cos(\theta)\cos(\phi)d\theta - r\sin(\theta)\sin(\phi)d\phi)\\ &\wedge (\sin(\theta)\sin(\phi)dr + r\cos(\theta)\sin(\phi)d\theta + r\sin(\theta)\cos(\phi)d\phi)\\ &\wedge (\cos(\theta)dr - r\sin(\theta)d\theta)\\ =&(r^2\sin^3(\theta)\cos^2(\phi) + r^2\sin(\theta)\cos^2(\theta)\cos^2(\phi) + r^2\sin^3(\theta)\sin^2(\phi)\\ &+ r^2\sin(\theta)\cos^2(\theta)\sin^2(\phi))dr \wedge d\theta \wedge d\phi\\ =&r^2(\sin^3(\theta) + \sin(\theta)\cos^2(\theta))dr \wedge d\theta \wedge d\phi\\ =&r^2\sin(\theta)dr \wedge d\theta \wedge d\phi.\end{aligned}$$

Next we show that this $\det J_\alpha = r^2\sin(\theta)$. By definition of the Jacobian matrix, we get:

$$\begin{aligned}\det J_\alpha =&\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{pmatrix}\\ =&\det\begin{pmatrix} \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi)\\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi)\\ \cos(\theta) & -r\sin(\theta) & 0\end{pmatrix}\\ =&r^2\sin(\theta)\cos^2(\theta)\cos^2(\phi) + r^2\sin^3(\theta)\sin^2(\phi) + r^2\sin^3(\theta)\cos^2(\phi)\\ &+ r^2\sin(\theta)\cos^2(\theta)\sin^2(\phi)\\ =&r^2\sin(\theta).\end{aligned}$$

Therefore

$$\alpha^*(dx \wedge dy \wedge dz) = r^2\sin(\theta)\ dr \wedge d\theta \wedge d\phi = (\det J_\alpha)dr \wedge d\theta \wedge d\phi$$

as claimed.

(b) To find $\alpha^*\omega$ we can use the result in (a). We get:

$$\begin{aligned}\alpha^*\omega =&(r^2\sin^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + r^2\cos^2(\theta))\alpha^*(dx \wedge dy \wedge dz)\\ =&(r^2\sin^2(\theta) + r^2\cos^2(\theta))r^2\sin(\theta)dr \wedge d\theta \wedge d\phi\\ =&r^4\sin(\theta)dr \wedge d\theta \wedge d\phi.\end{aligned}$$

3. Let

$$\omega = (x^2 + y^2)\,(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy)$$

on $\mathbb{R}^3$, and $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the cylindrical transformation

$$\alpha(r, \theta, w) = (r\cos\theta, r\sin\theta, w).$$

(a) Find $\alpha^*\omega$.

(b) Show by explicit calculation that $d(\alpha^*\omega) = \alpha^*(d\omega)$.

Solution.

(a) To simplify the calculation of the pullback, let us first calculate the pullback of the basic two-forms:

$$\begin{aligned}\alpha^*(dy \wedge dz) =& (\sin(\theta)dr + r\cos(\theta)d\theta) \wedge dw \\ =& \sin(\theta)dr \wedge dw + r\cos(\theta)d\theta \wedge dw,\end{aligned}$$

$$\begin{aligned}\alpha^*(dz \wedge dx) =& dw \wedge (\cos(\theta)dr - r\sin(\theta)d\theta) \\ =& -\cos(\theta)dr \wedge dw + r\sin(\theta)d\theta \wedge dw,\end{aligned}$$

and

$$\begin{aligned}\alpha^*(dx \wedge dy) =& (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\ =& r\cos^2(\theta)dr \wedge d\theta + r\sin^2(\theta)dr \wedge d\theta \\ =& r dr \wedge d\theta.\end{aligned}$$

We also observe that

$$\alpha^*(x^2 + y^2) = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2.$$

Putting this together, we get:

$$\begin{aligned}\alpha^*\omega =& \alpha^*(x^2 + y^2)\alpha^*\left(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy\right) \\ =& r^2(r\cos(\theta)(\sin(\theta)dr \wedge dw + r\cos(\theta)d\theta \wedge dw) \\ &+ r\sin(\theta)(-\cos(\theta)dr \wedge dw + r\sin(\theta)d\theta \wedge dw) + rw dr \wedge d\theta) \\ =& r^3(r d\theta \wedge dw + w dr \wedge d\theta).\end{aligned}$$

(b) On the one hand, we calculated in (a) the pullback $\alpha^*\omega = r^3(rd\theta \wedge dw + wdr \wedge d\theta)$.We can calculate its exterior derivative:

$$\begin{aligned}d(\alpha^*\omega) =& d(r^4) \wedge d\theta \wedge dw + d(r^3 w) \wedge dr \wedge d\theta \\ =& (4r^3 + r^3)dr \wedge d\theta \wedge dw. \\ =& 5r^3 dr \wedge d\theta \wedge dw.\end{aligned}$$

On the other hand, we can calculate first the exterior derivative of ω. We get:

$$\begin{aligned}d\omega =& d((x^2 + y^2)x) \wedge dy \wedge dz + d((x^2 + y^2)y) \wedge dz \wedge dx + d((x^2 + y^2)z) \wedge dx \wedge dy \\ =& ((3x^2 + y^2) + (3y^2 + x^2) + (x^2 + y^2))dx \wedge dy \wedge dz \\ =& 5(x^2 + y^2)dx \wedge dy \wedge dz.\end{aligned}$$

We then calculate its pullback:

$$\begin{aligned}\alpha^*(d\omega) =& 5\alpha^*(x^2+y^2)\alpha^*(dx \wedge dy \wedge dz)\\ =& 5r^2(\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \wedge dw\\ =& 5r^2(r\cos^2(\theta) + r\sin^2(\theta))dr \wedge d\theta \wedge dw\\ =& 5r^3\ dr \wedge d\theta \wedge dw.\end{aligned}$$

We conclude that

$$d(\alpha^*\omega) = \alpha^*(d\omega),$$

as claimed.

4. Let

$$\omega = ze^{xy}\ dx \wedge dy$$

on $\mathbb{R}^3$, and $\phi : (\mathbb{R}_{\neq 0})^2 \to \mathbb{R}^3$ with:

$$\phi(u,v) = \left(\frac{u}{v}, \frac{v}{u}, uv\right).$$

Find $\phi^*\omega$.

Solution. We calculate the pullback:

$$\begin{aligned}\phi^*\omega =& uve^{\frac{u}{v}\frac{v}{u}}\left(\frac{1}{v}du - \frac{u}{v^2}dv\right) \wedge \left(-\frac{v}{u^2}du + \frac{1}{u}dv\right)\\ =& uve\left(\frac{1}{uv} - \frac{1}{uv}\right)du \wedge dv\\ =& 0.\end{aligned}$$

5. Show that the pullback of an exact form is always exact.

Solution. Let ω be an exact k-form on $U \subseteq \mathbb{R}^n$, and let $\phi : V \to U$ be a smooth function for $V \subseteq \mathbb{R}^m$. We want to prove that the pullback of ω is exact.

Since ω is exact, we know that there exists a $(k-1)$-form η on U such that $\omega = d\eta$. Then

$$\phi^*\omega = \phi^*(d\eta) = d(\phi^*\eta),$$

where we used the fact that the pullback commutes with the exterior derivative (see Lemma 4.7.4). Therefore, the k-form $\phi^*\omega$ on V is the exterior derivative of a $(k-1)$-form $\phi^*\eta$ on V, and hence it is exact.

6. Let ω be a k-form on $\mathbb{R}^n$, and $Id : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function defined by $Id(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Show that

$$Id^*\omega = \omega.$$

Solution. We can write a general k-form on $\mathbb{R}^n$ as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some smooth functions $f_{i_1 \cdots i_k} : \mathbb{R}^n \to \mathbb{R}$. We know that $Id^*(dx_i) = dx_i$ for all $i \in \{1, \ldots, n\}$, by definition of the identity map. Similarly, for any function $f : \mathbb{R}^n \to \mathbb{R}$,

$Id^*f = f$. As a result, we get:

$$\begin{aligned}Id^*\omega =& \sum_{1\leq i_1<\cdots<i_k\leq n} Id^*(f_{i_1\cdots i_k})Id^*(dx_{i_1}) \wedge \cdots \wedge Id^*(dx_{i_k}) \\ =& \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ =& \omega.\end{aligned}$$

7. Let ω be a k-form on $U \subseteq \mathbb{R}^n$. Let $\phi : V \to U$ and $\alpha : W \to V$ be smooth functions, with $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^\ell$. Show that

$$(\phi \circ \alpha)^*\omega = \alpha^*(\phi^*\omega).$$

In other words, it doesn't matter whether we pullback in one or two steps in the chain of maps

$$W \xrightarrow{\alpha} V \xrightarrow{\phi} U.$$

Solution. We can write a general k-form on U as

$$\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some smooth functions $f_{i_1\cdots i_k} : U \to \mathbb{R}$. On the one hand, the pullback by $\phi \circ \alpha$ is

$$(\phi \circ \alpha)^*\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} (\phi \circ \alpha)^*(f_{i_1\cdots i_k})(\phi \circ \alpha)^*(dx_{i_1}) \wedge \cdots \wedge (\phi \circ \alpha)^*(dx_{i_k}).$$

On the other hand, pulling back in two steps, we get:

$$\alpha^*(\phi^*\omega) = \sum_{1\leq i_1<\cdots<i_k\leq n} \alpha^*(\phi^* f_{i_1\cdots i_k})\alpha^*(\phi^* dx_{i_1}) \wedge \cdots \wedge \alpha^*(\phi^* dx_{i_k}).$$

So all we have to show is that

$$(\phi \circ \alpha)^* f = \alpha^*(\phi^* f)$$

for any smooth function $f : U \to \mathbb{R}$, and

$$(\phi \circ \alpha)^* dx_i = \alpha^*(\phi^* dx_i)$$

for any $i \in \{1, \ldots, n\}$.

First, for any function $f : U \to \mathbb{R}$,

$$(\phi \circ \alpha)^* f = f \circ \phi \circ \alpha,$$

while

$$\alpha^*(\phi^* f) = \alpha^*(f \circ \phi) = f \circ \phi \circ \alpha.$$

Thus

$$(\phi \circ \alpha)^* f = \alpha^*(\phi^* f).$$

As for the basic one-forms, let us introduce further notation for the maps $\phi : V \to U$ and $\alpha : W \to V$. Let us write $\mathbf{z} = (z_1, \ldots, z_\ell)$ for coordinates on W; $\mathbf{y} = (y_1, \ldots, y_m)$ for coordinates on V; and $\mathbf{x} = (x_1, \ldots, x_n)$ for coordinates on U. We write $\phi(\mathbf{y}) = (\phi_1(\mathbf{y}, \ldots, \phi_n(\mathbf{y}))$, and $\alpha(\mathbf{z}) = (\alpha_1(\mathbf{z}), \ldots, \alpha_m(\mathbf{z}))$. Then, we have:

$$\phi^* dx_i = \sum_{a=1}^{m} \frac{\partial \phi_i}{\partial y_a} dy_a,$$

and

$$\alpha^*(\phi^* dx_i) = \sum_{b=1}^{\ell} \sum_{a=1}^{m} \frac{\partial \phi_i}{\partial y_a}\Big|_{\mathbf{y}=\alpha(\mathbf{z})} \frac{\partial \alpha_a}{\partial z_b} dz_b.$$

On the other hand, if we pullback by the composition of the maps, we get:

$$(\phi \circ \alpha)^* dx_i = \sum_{b=1}^{\ell} \frac{\partial (\phi \circ \alpha)_i}{\partial z_b} dz_b.$$

But the equality

$$\frac{\partial (\phi \circ \alpha)_i}{\partial z_b} = \sum_{a=1}^{m} \frac{\partial \phi_i}{\partial y_a}\Big|_{\mathbf{y}=\alpha(\mathbf{z})} \frac{\partial \alpha_a}{\partial z_b}$$

is precisely the chain rule for multi-variable functions (written in terms of composition of functions). Thus we conclude that

$$\alpha^*(\phi^* dx_i) = (\phi \circ \alpha)^* dx_i.$$

Putting all of this together, we conclude that

$$(\phi \circ \alpha)^* \omega = \alpha^*(\phi^* \omega),$$

which is the statement of the question.

4.8 Hodge star

In this section we introduce one last operator on differential forms, called the "Hodge star". While we will not really use it in this course, it is an integral part of the theory of differential forms, so it is worth being introduced to it briefly. The Hodge star can be used to recover the Laplacian operator in the language of vector calculus.

Objectives

You should be able to:

- Define the Hodge star operator in $\mathbb{R}^n$.
- Determine the Hodge star of zero-, one-, two-, and three-forms in $\mathbb{R}^3$.
- Relate to the Laplacian operator in vector calculus.

4.8.1 The Hodge star

The Hodge star is an operator that provides some sort of duality between k-forms and $(n-k)$-forms in $\mathbb{R}^n$. It is easiest to define it in terms of the basic k-forms from Definition 4.1.5, and then extend to general differential forms by applying it to each summand.

Definition 4.8.1 The Hodge star dual of a k-form in $\mathbb{R}^n$. Let $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ be a basic k-form on $\mathbb{R}^n$. Then the **Hodge star dual of** ω, which is denoted by $\star\omega$,[1] is the unique basic $(n-k)$-form with the property:

$$\omega \wedge \star\omega = dx_1 \wedge \cdots \wedge dx_n.$$

To define the Hodge star dual of a general k-form on $U \subseteq \mathbb{R}^n$, we apply the Hodge star to each summand. More precisely, if

$$\eta = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \cdots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is a k-form on U, then its **Hodge star dual** $*\eta$ is the $(n-k)$-form given by

$$\star\eta = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \cdots i_k} \, \star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}).$$

◊

To make sense of this definition, let us look at the Hodge star action on the basic k-forms for low-dimensional space.

Example 4.8.2 The action of the Hodge star in $\mathbb{R}$. There are only two basic k-forms in $\mathbb{R}$, namely the zero-form 1 and the one-form dx. From the definition, we want $1 \wedge \star 1 = dx$ and $dx \wedge \star dx = dx$, from which we conclude that:

$$\star 1 = dx, \qquad \star dx = 1.$$

The Hodge star thus provides a duality between zero-forms and one-forms in $\mathbb{R}$. □

Example 4.8.3 The action of the Hodge star in $\mathbb{R}^2$. It becomes a little more interesting in $\mathbb{R}^2$. The basic forms are the zero-form 1, the one-forms dx and dy, and the two-form $dx \wedge dy$. From the definition, we want $1 \wedge \star 1 = dx \wedge dy$, $dx \wedge \star dx = dx \wedge dy$, $dy \wedge \star dy = dx \wedge dy$, and $(dx \wedge dy) \wedge \star(dx \wedge dy) = dx \wedge dy$. We conclude that

$$\begin{aligned} \star 1 &= dx \wedge dy, & \star(dx \wedge dy) &= 1, \\ \star dx &= dy, & \star dy &= -dx. \end{aligned}$$

It thus provides a duality between zero-forms and two-forms in $\mathbb{R}^2$, and a "self-duality" for one-forms. Note that the sign is important here for the action on the basic one-forms. □

Example 4.8.4 The action of the Hodge star in $\mathbb{R}^3$. Things become even more interesting in $\mathbb{R}^3$. The basic forms are the zero-form 1, the one-forms dx, dy, dz, the two-forms

[1] This standard notation should not be confused with the pullback of a differential form; those are very different things.

$dy \wedge dz, dz \wedge dx, dx \wedge dy$, and the three-form $dx \wedge dy \wedge dz$. From the definition, we get that:

$$\star 1 = dx \wedge dy \wedge dz, \qquad \star(dx \wedge dy \wedge dz) = 1,$$
$$\star dx = dy \wedge dz, \qquad \star dy = dz \wedge dx, \qquad \star dz = dx \wedge dy,$$
$$\star(dy \wedge dz) = dx, \qquad \star(dz \wedge dx) = dy, \qquad \star(dx \wedge dy) = dz.$$

Thus, in $\mathbb{R}^3$, it provides a duality between zero-forms and three-forms, and between one-forms and two-forms. □

Example 4.8.5 An example of the Hodge star action in $\mathbb{R}^3$. Consider the two-form $\omega = xyz\ dy \wedge dz + e^x\ dx \wedge dy$. Its Hodge star dual is the one-form:

$$\begin{aligned}\star\omega =& xyz\ \star(dy \wedge dz) + e^x\ \star(dx \wedge dy)\\ =& xyz\ dx + e^x\ dz.\end{aligned}$$

□

The action of the Hodge star in $\mathbb{R}^3$ naturally justifies our dictionary to translate between k-forms in $\mathbb{R}^3$ and vector calculus objects in Table 4.1.11. Indeed, let $\omega = f\ dx + g\ dy + h\ dz$ be a one-form on $\mathbb{R}^3$. Then its Hodge dual is the two-form

$$\star\omega = f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy.$$

This is why we used this particular choice for the basic two-forms in Table 4.1.11; it's because this is what one gets through Hodge duality, which identifies one-forms and two-forms in $\mathbb{R}^3$.

In fact, what this means is that we really only needed the first two lines in Table 4.1.11. Indeed, we can always transform a two-form into a one-form by taking its Hodge dual, and a three-form into a zero-form. So, in the end, all that we need to establish a dictionary between differential forms and vector calculus objects is to say that zero-forms are functions, and one-forms correspond to vector fields.

For instance, if $\mathbf{F}$ is the vector field associated to a one-form ω, we could have defined the curl $\boldsymbol{\nabla} \times \mathbf{F}$ to be the vector field associated to the one-form $\star d\omega$. That is,

$$\omega \leftrightarrow \mathbf{F} \qquad \star\, d\omega \leftrightarrow \boldsymbol{\nabla} \times \mathbf{F}.$$

Similarly, we could have defined the divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$ to be the function given by $\star d \star \omega$. That is,

$$\omega \leftrightarrow \mathbf{F} \qquad \star\, d \star \omega \leftrightarrow \boldsymbol{\nabla} \cdot \mathbf{F}.$$

We could translate all vector calculus identities in Section 4.4 using the Hodge star, but in the end, as far as we are concerned in this course, this is just a fancier way of saying the same thing. :-)

Example 4.8.6 Maxwell's equations using differential forms (optional). Recall from Example 4.4.7 the statement of Maxwell's equations, which form the foundations of electromagnetism. They can be written in terms of the electric vector field $\mathbf{E}$ and the magnetic vector field $\mathbf{B}$ on $\mathbb{R}^3$ as follows (I am now using units with $c = 1$ as is standard in modern physics, and I have rescaled the electric charge ρ and the electric current density $\mathbf{J}$ to absorb the factor of 4π):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho,$$

$$\begin{aligned}\nabla \cdot \mathbf{B} =&0,\\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} =&0,\\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} =&\mathbf{J}.\end{aligned}$$

It turns out that there is a very nice way of rewriting Maxwell's equations using differential forms, which makes them manifestly relativistic (i.e. consistent with special relativity). Moreover, this reformulation works in any number of dimensions! It defines the natural generalization of Maxwell's equations to higher-dimensional spacetimes, which is useful in physics theories like string theory.

To write Maxwell's equations in this form, we need to consider them as living on spacetime, i.e. $\mathbb{R}^4$, with coordinates (t, x, y, z). We first construct a two-form F on $\mathbb{R}^4$ that combines the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ as follows:

$$F = B_x\ dy \wedge dz + B_y\ dz \wedge dx + B_z\ dx \wedge dy + E_x\ dx \wedge dt + E_y\ dy \wedge dt + E_z\ dz \wedge dt.$$

We also construct a three-form which combines the electric current $\mathbf{J}$ and the electric charge ρ as:

$$J = \rho\ dx \wedge dy \wedge dz - j_x\ dt \wedge dy \wedge dz - j_y\ dt \wedge dz \wedge dx - j_z\ dt \wedge dx \wedge dy.$$

Using these definitions, and the definition of the Hodge star on $\mathbb{R}^4$,[2] we can rewrite Maxwell's equations neatly as the following two equations (you can check this!):

$$\begin{aligned} dF =&0,\\ d \star F =&J.\end{aligned}$$

Isn't that neat? The first equation is simply saying that the two-form F is closed. The second equation is the "source equation"; if there is no source (i.e. $J = 0$), it is simply saying that the two-form $\star F$ is also closed. Moreover, not only is this formulation nice and clean, but it is manifestly Lorentz invariant (as it is formulated in four-dimensional space time).

Furthermore, this formulation of Maxwell's equation naturally generalizes to any number of dimensions. In $\mathbb{R}^n$, F remains a two-form, but J becomes an $(n-1)$-form. Then $\star F$ is an $(n-2)$-form, and the two equations make sense in $\mathbb{R}^n$. As mentioned above, this higher-dimensional generalization is useful in modern physical theories such as string theory. This is an example of the power of the formalism of differential forms! □

[2] To be precise, we need to modify the definition of the Hodge star operator a little bit here. The reason is that spacetime is $\mathbb{R}^4$, but with a different definition of length as for standard Euclidean space. It is called Minkowski spacetime; the definition of length is given by a metric, and in this case the metric is Lorentzian, which is a bit different from the standard Euclidean metric (it has an extra sign). The reason is that the fourth dimension of time behaves somewhat differently from the three space dimensions, which is reflected in this choice of Lorentzian metric. In the end, the correct definition of the Hodge star operator on Minkowski spacetime $\mathbb{R}^4$ is the operator $\star$ that acts on basic k-form as

$$\omega \wedge \star\omega = \pm dt \wedge dx \wedge dy \wedge dz,$$

with a plus sign whenever ω does not contain dt, and a minus sign whenever ω contains dt.

4.8.2 The Hodge star and the Laplacian

We can combine the Hodge star with the exterior derivative to define a new operation on differential forms, called the "Laplace-Beltrami operator".

Definition 4.8.7 The codifferential and the Laplace-Beltrami operator. Let ω be a k-form on $U \subseteq \mathbb{R}^n$. We define the **codifferential**, which is denoted by δ, as the operator acting on ω as:

$$\delta\omega = (-1)^k \star d \star \omega.$$

We define the **Laplace-Beltrami operator**, denoted by Δ, as the operator acting on ω as:

$$\Delta\omega = -(d\delta + \delta d)\omega,$$

where d is the exterior derivative. ◊

This looks very fancy, but it is just the natural generalization of the Laplacian of a function to differential forms, as we now see.

Lemma 4.8.8 The Laplace-Beltrami operator and the Laplacian of a function. *Let f be a zero-form (a function) on $U \subseteq \mathbb{R}^3$. Then*

$$\Delta f = \nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

*where $\nabla^2 f$ is called the **Laplacian of** f.*

Proof. First, we notice that if f is a function,

$$\delta f = \star d \star f = 0,$$

since $\star f$ is a three-form, and hence its exterior derivative vanishes in $\mathbb{R}^3$. Thus

$$\Delta f = -\delta df = \star d \star df.$$

We now translate into the language of vector calculus. The vector field associated to the one-form df is the gradient $\boldsymbol{\nabla} f$. Its Hodge dual is a two-form, and taking its exterior derivative means that we take the divergence of this vector field. So the result is that

$$\Delta f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f.$$

Expanding in coordinates (x, y, z), we get:

$$\Delta f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

which we write as the Laplacian $\nabla^2 f$ of the function f. ■

The Laplace-Beltrami operator applied to a one-form in $\mathbb{R}^3$ gives rise to another operation in vector calculus, called the "Laplacian of a vector field".

Lemma 4.8.9 The Laplace-Beltrami operator and the Laplacian of a vector field. *Let ω be a one-form on $U \subseteq \mathbb{R}^3$, and $\mathbf{F}$ be its associated vector field. Then the vector field* *associated to the one-form $\Delta\omega$ is the* ***Laplacian of the vector field*** $\mathbf{F}$*, which is denoted by* $\nabla^2\mathbf{F}$*. It acts on the vector field as*

$$\nabla^2\mathbf{F} = \frac{\partial^2\mathbf{F}}{\partial x^2} + \frac{\partial^2\mathbf{F}}{\partial y^2} + \frac{\partial^2\mathbf{F}}{\partial z^2},$$

and satisfies the identity

$$\nabla^2\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}).$$

Proof. If ω is a one-form, by definition

$$\Delta\omega = -d\delta\omega - \delta d\omega = d\star d\star\omega - \star d\star d\omega,$$

which is also a one-form. Let us now extract its associated vector field. Let $\mathbf{F}$ be the vector field associated to ω.

We look at the first term on the right-hand-side. $\star\omega$ is a two-form associated to $\mathbf{F}$. $d\star\omega$ then takes the divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$. $\star d\star\omega$ maps this to a zero-form, and then $d\star d\star\omega$ take the gradient of the resuling zero-form. The result is that the vector field associated to the one-form $d\star d\star\omega$ is

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}).$$

Let us now look at the second term on the right-hand-side. $d\omega$ takes the curl $\boldsymbol{\nabla}\times\mathbf{F}$. $\star d\omega$ then maps it to a one-form associated to the vector field $\boldsymbol{\nabla}\times\mathbf{F}$. $d\star d\omega$ then takes the curl again, $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})$, and finaly $\star d\star d\omega$ maps it back to a one-form. The result is that the vector field associated to the one-form $\star d\star d\omega$ is

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}).$$

We conclude that the vector field associated to the one-form $\Delta\omega$ is

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}),$$

which we can take as the definition of the Laplacian of the vector field $\mathbf{F}$. To show that it takes the form

$$\nabla^2\mathbf{F} = \frac{\partial^2\mathbf{F}}{\partial x^2} + \frac{\partial^2\mathbf{F}}{\partial y^2} + \frac{\partial^2\mathbf{F}}{\partial z^2},$$

one only needs to do an explicit calculation in $\mathbb{R}^3$, see Exercise 4.8.4.5. ∎

4.8.3 Two more vector calculus identities

To end this section, let us prove two more vector calculus identities, this time involving the Laplacian of a function.

Lemma 4.8.10 Vector calculus identities, part 5. *Let f and g be smooth functions on $U\subseteq\mathbb{R}^3$. Then it satisfies the identities:*

1. $$\nabla^2(fg) = f\nabla^2 g + 2\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + g\nabla^2 f,$$

2.

$$\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f) = f \nabla^2 g - g \nabla^2 f.$$

Proof. These two identities follow from the graded product rule for the exterior derivative.

$$\begin{aligned} dF =& 0, \\ d \star F =& J. \end{aligned}$$

We start with the first one:

$$\begin{aligned} \nabla^2(fg) =& \Delta(fg) \\ =& \star d \star d(fg) \\ =& \star d \star (g\ df + f\ dg) \\ =& \star d(g\ \star df + f\ \star dg) \\ =& \star (dg \wedge \star df + g\ d \star df + df \wedge \star dg + f d \star dg) \\ =& \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f + g \nabla^2 f + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \nabla^2 g \\ =& f \nabla^2 g + 2 \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + g \nabla^2 f. \end{aligned}$$

In the proof we used the fact that $\nabla^2 f = \star d \star df$ as in the proof of Lemma 4.8.8.

For the second identity, we get the following:

$$\begin{aligned} \boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f) &= \star d \star (f dg - g df) \\ &= \star d(f \star dg - g \star df) \\ &= \star (df \wedge \star dg + f d \star dg - dg \wedge \star df - g d \star df) \end{aligned}$$

To proceed we need to use a result which we haven't proved. For any two k-forms ω and η on $\mathbb{R}^3$, there's a general result that says that

$$\omega \wedge \star \eta = \star \omega \wedge \eta.$$

Note that it is important that ω and η are both k-forms (same k), otherwise it wouldn't apply. It it not difficult to prove this statement, but since we do not need it anywhere else, we leave the proof as an exercise (see Exercise 4.8.4.4).

Now in our previous expression we had the terms $df \wedge \star dg$ and $-dg \wedge \star df$. Since df and dg are both one-forms,

$$df \wedge \star dg = \star df \wedge dg,$$

and these two terms cancel out. Thus

$$\begin{aligned} \boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f) =& f(\star d \star dg) - g(\star d \star df) \\ =& f \nabla^2 g - g \nabla^2 f. \end{aligned}$$

■

4.8.4 Exercises

1. Let ω be the two-form

$$\omega = xy\ dy \wedge dz + xyz\ dz \wedge dx + y\ dx \wedge dy$$

on $\mathbb{R}^3$. Find $\star\omega$.

Solution. We calculate the one-form $\star\omega$ using the action of the Hodge star on basic two-forms in $\mathbb{R}^3$:

$$\begin{aligned}\star\omega =& xy \star (dy \wedge dz) + xyz \star (dz \wedge dx) + y \star (dx \wedge dy)\\ =& xy\ dx + xyz\ dy + y\ dz.\end{aligned}$$

2. For any k-form ω on $\mathbb{R}^n$, show that

$$\star \star \omega = (-1)^{k(n-k)}\omega.$$

Solution. We only need to prove the statement for basic k-forms as by definition of the action of the Hodge star operator in Definition 4.8.1 it will then follow for all k-forms.

Let α be a basic k-form on $\mathbb{R}^n$. By definition of the Hodge star, we know that $\star\alpha$ is the unique basic $(n-k)$-form such that

$$\alpha \wedge \star\alpha = dx_1 \wedge \cdots \wedge dx_n.$$

Now consider the basic $(n-k)$-form $\star\alpha$. By definition of the Hodge star, $\star\star\alpha$ will be the unique basic k-form such that

$$\star\alpha \wedge \star\star\alpha = dx_1 \wedge \cdots \wedge dx_n.$$

As the right-hand-side for both equations is the same, we get

$$\alpha \wedge \star\alpha = \star\alpha \wedge \star\star\alpha.$$

Using graded commutativity of the wedge product as in Lemma 4.2.6, we can rewrite the right-hand-side as:

$$\alpha \wedge \star\alpha = (-1)^{k(n-k)} \star\star\alpha \wedge \star\alpha.$$

Finally, given $\star\alpha$, we know that the left-hand-side uniquely defines α (by definition of the Hodge star), while the right-hand-side uniquely deefines $\star\star\alpha$ (again by definition of the Hodge star), and therefore we must have

$$\alpha = (-1)^{k(n-k)} \star\star\alpha.$$

3. Let ω and η be one-forms on $\mathbb{R}^n$, and $\mathbf{F}$ and $\mathbf{G}$ be the associated vector fields. Show that

$$\star(\omega \wedge \star\eta) = \mathbf{F} \cdot \mathbf{G}.$$

Solution. We can write the one-forms ω and η as:

$$\omega = \sum_{i=1}^n f_i dx_i,$$

$$\eta = \sum_{i=1}^{n} g_i dx_i,$$

for smooth functions $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$. Since $\star\eta$ is an $(n-1)$-form, $\omega \wedge \star\eta$ is an n-form on $\mathbb{R}^n$. It then follows that the only non-vanishing terms in $\omega \wedge \star\eta$ are those of the form $dx_i \wedge \star dx_i$, as all other "cross-terms", i.e. terms of the form $dx_i \wedge \star dx_j$ with $i \neq j$, will necessarily vanish since $\star dx_j$ necessarily contains a dx_i. Thus we get:

$$\begin{aligned}\omega \wedge \star\eta &= \sum_{i=1}^{n} f_i g_i dx_i \wedge \star dx_i \\ &= \left(\sum_{i=1}^{n} f_i g_i\right) dx_1 \wedge \cdots \wedge dx_n,\end{aligned}$$

where we used the definition of the Hodge star for basic one-forms. Finally, since $\star(dx_1 \wedge \cdots dx_n) = 1$, we get:

$$\star(\omega \wedge \star\eta) = \sum_{i=1}^{n} f_i g_i \star (dx_1 \wedge \cdots \wedge dx_n) = \sum_{i=1}^{n} f_i g_i = \mathbf{F} \cdot \mathbf{G},$$

where the last equality is for the associated vector fields $\mathbf{F} = (f_1, \ldots, f_n)$ and $\mathbf{G} = (g_1, \ldots, g_n)$.

4. Let ω and η be k-forms on $\mathbb{R}^n$. Show that

$$\omega \wedge \star\eta = (-1)^{k(n-k)} \star \omega \wedge \eta.$$

Note that it is important that ω and η are both k-forms (same k), otherwise this property wouldn't apply.

Solution. The proof is similar in spirit to the solution of the previous problem. Since ω and η are both k-forms on $\mathbb{R}^n$, we can write both as linear combinations of basic k-forms in $\mathbb{R}^n$:

$$\begin{aligned}\omega &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \\ \eta &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},\end{aligned}$$

for smooth functions $f_{i_1 \cdots i_k}, g_{i_1 \cdots i_k} : \mathbb{R}^n \to \mathbb{R}$. Then

$$\star\eta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} \star (dx_{i_1} \wedge \cdots \wedge dx_{i_k}).$$

But $\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ is an $(n-k)$-form on $\mathbb{R}^n$, and thus $\omega \wedge \star\eta$ is an n-form in $\mathbb{R}^n$. It follows that the only terms in $\omega \wedge \star\eta$ that will be non-vanishing are those involving the wedge product of a basic k-form with its own Hodge star dual, as all other "cross-terms" will necessarily involve the wedge products of repeated dx's which trivially vanish. So we get:

$$\omega \wedge \star\eta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} g_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}).$$

Similarly, we have:

$$\begin{aligned}\star\omega \wedge \eta =& \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k} g_{i_1\cdots i_k} \star (dx_{i_1}\wedge\cdots\wedge dx_{i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\\ =&(-1)^{k(n-k)}\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k} g_{i_1\cdots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge\star(dx_{i_1}\wedge\cdots\wedge dx_{i_k}),\end{aligned}$$

where we used graded commutativity of the wedge product, Lemma 4.2.6. We thus conclude that

$$\omega\wedge\star\eta = (-1)^{k(n-k)}\star\omega\wedge\eta.$$

5. Let $\mathbf{F} = (f_1, f_2, f_3)$ be a smooth vector field in $\mathbb{R}^3$. Show that

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) = \frac{\partial^2\mathbf{F}}{\partial x^2} + \frac{\partial^2\mathbf{F}}{\partial y^2} + \frac{\partial^2\mathbf{F}}{\partial z^2}.$$

This is the definition of the Laplacian of the vector field $\nabla^2\mathbf{F}$ as in Lemma 4.8.9.

Solution. This is just an explicit and rather painful calculation. Let us do it step-by-step. First,

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

Thus

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) = \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x\partial y} + \frac{\partial^2 f_3}{\partial x\partial z}, \frac{\partial^2 f_1}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial y\partial z}, \frac{\partial^2 f_1}{\partial z\partial x} + \frac{\partial^2 f_2}{\partial z\partial y} + \frac{\partial^2 f_3}{\partial z^2}\right).$$

Next, we move on to the curl. First, we have

$$\boldsymbol{\nabla}\times\mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

Taking the curl again, we get:

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) =& \left(\frac{\partial^2 f_2}{\partial y\partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z\partial x}, \frac{\partial^2 f_3}{\partial z\partial y} - \frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_1}{\partial x\partial y},\right.\\ &\left.\frac{\partial^2 f_1}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_2}{\partial y\partial z}\right)\end{aligned}$$

Putting these two calculations together, and using the fact that partial derivatives commute by Clairaut's theorem (since the vector fields are assumed to be smooth), we get:

$$\begin{aligned}&\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})\\ =&\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}, \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}, \frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right)\\ =&\frac{\partial^2\mathbf{F}}{\partial x^2} + \frac{\partial^2\mathbf{F}}{\partial y^2} + \frac{\partial^2\mathbf{F}}{\partial z^2}.\end{aligned}$$

6. In this problem we prove the statement in Example 4.8.6 about Maxwell's equations.

(a) Write down the action of the Hodge star operator on basic k-forms in Minkowksi $\mathbb{R}^4$ (see Footnote 4.8.2).

(b) Using your result in part (a), show that the two equations

$$\begin{aligned} dF &= 0, \\ d \star F &= J, \end{aligned}$$

with F and J defined in Example 4.8.6, reproduce Maxwell's equations.

Chapter 5

Integrating two-forms: surface integrals

We study how two-forms can be integrated along surfaces, which leads us to introduce the concept of surface integrals (also called “flux integrals”).

5.1 Integrating zero-forms and one-forms

Before we move on to two-forms, for completeness we start by defining how zero-forms can be integrated over oriented points, and review how one-forms can be integrated over oriented curves. We highlight the main steps of the construction, setting up the stage for the development of surface integrals.

Objectives

You should be able to:

- Define the integral of a zero-form over oriented points.
- Rephrase the Fundamental Theorem of line integrals in terms of oriented integrals of one- and zero-forms.

5.1.1 General strategy

Let us start by summarizing step-by-step how we are developing our theory of integration. Suppose that we want to define integration of k-forms over k-dimensional spaces:

1. We define the orientation of a closed bounded region $D \subset \mathbb{R}^k$ (such as a closed interval in $\mathbb{R}$), and the notion of canonical orientation. We define the induced orientation on the boundary of the region ∂D.
2. We define the integral of a k-form over an oriented region D in terms of standard multiple integrals from calculus. To define the integral over a bounded region that has many components, we sum over the integrals on each components.

3. We define the notion of a parametric space, which maps a region D to a k-dimensional subspace $S \subset \mathbb{R}^n$, with $n > k$. We show that the parametrization induces an orientation on S.

4. To define the integral of a k-form in $\mathbb{R}^n$ on the subspace $S \subset \mathbb{R}^n$ with a choice of orientation, we use the parametrization to pullback the k-form to the region $D \subset \mathbb{R}^k$, and then we integrate as in (2) using standard calculus.

5. We show that the integral is invariant under orientation-preserving reparametrizations and changes sign under orientation-reversing reparametrizations, thus ensuring that our theory is oriented and reparametrization-invariant. We conclude that the integral is defined geometrically in terms of the subspace S and a choice of orientation.

6. As a last step, we study what happens in the case of an exact k-form: this leads to Stokes' Theorem, which is the higher dimensional generalization of the Fundamental Theorem of Calculus and the Fundamental Theorem of line integrals.

This summarizes the conceptual steps to construct our theory of integration. This is pretty much exactly what we did for line integrals in Chapter 3; we will review these steps below. But before we do that, let us apply this strategy to construct integrals of zero-forms.

5.1.2 Integrating zero-forms over oriented points

We consider first the very simple and particular case of zero-forms. We would like to integrate a zero-form over a zero-dimensional space. What is a zero-dimensional space? It is just a point (or a union of points). But let us construct the theory step-by-step.

STEP 1. We first need to define the orientation of a point.

Definition 5.1.1 Oriented points. Pick a point $a \in \mathbb{R}^n$. The **orientation of a point** $a \in \mathbb{R}^n$ is given by a choice of $+$ or $-$. We write an oriented point as $(a, +)$ or $(a, -)$. The **canonical orientation** is $+$. ♢

According to our recipe, we should talk about the induced orientation on the boundary, but a point has no boundary, so this step is meaningless in this case.

STEP 2, 3, 4, 5. The next step is to define the "integral" of a zero-form on an oriented point in $\mathbb{R}$. Then we would define "parametrizations for points in higher-dimensional spaces", but this is all rather trivial here, so we can do steps 2 to 5 all at the same time and simply define the integral of a zero-form on an oriented point in $\mathbb{R}^n$ directly.

Recall that a zero-form is just a function f. In this case, the integral is defined very simply by just evaluating the function at the point.

Definition 5.1.2 Integral of a zero-form over an oriented point. Let f be a zero-form on an open subset $U \subseteq \mathbb{R}^n$ and $(P, \pm)$ a point in U with a choice of orientation. We define the **integral of f on** $(P, \pm)$ by:

$$\int_{(P,\pm)} f = \pm f(P).$$

In other words, we just evaluate the function at $P \in \mathbb{R}^n$, and multiply by the sign corresponding to the chosen orientation.

To define the integral over a set of oriented points, we sum up the integrals over each point separately. ♢

For instance, in this language we can define the integral of a function f over two oriented points $\{(P_0, -), (P_1, +)\}$ as:

$$\int_{\{(P_0,-),(P_1,+)\}} f = f(P_1) - f(P_0).$$

And that's basically it, as far as zero-forms go. There's no question of parametrization here, and the integral is clearly oriented by definition. There's no step 6, as there is no such thing as an exact zero-form.

Example 5.1.3 Integral of a zero-form at points. Consider the function $f(x, y, z) = xy + z$ on $\mathbb{R}^3$. Pick the two points $P_0 = (0, 0, 0)$ and $P_1 = (1, 1, 1)$. Let's give P_0 a negative orientation, and P_1 a positive orientation. Then

$$\int_{\{(P_0,-),(P_1,+)\}} f = f(P_1) - f(P_0) = f(1, 1, 1) - f(0, 0, 0) = 2.$$

□

5.1.3 Integrating one-forms over oriented curves

We move to the case of one-forms, which was already covered in Chapter 3. Here we review the construction to highlight how it fits within our general strategy.

STEP 1. We start by considering a closed interval $[a, b] \in \mathbb{R}$. We define its orientation as in Definition 3.1.3. We also define an induced orientation on the boundary.

Definition 5.1.4 The orientation of an interval. We define the **orientation of an interval** $[a, b] \subset \mathbb{R}$ to be a choice of direction. By $[a, b]_+$, we mean the interval $[a, b]$ with the orientation of increasing real numbers (from a to b), and by $[a, b]_-$ we denote the same interval but with the orientation of decreasing real numbers (from b to a). We define the **canonical orientation** to be the orientation of increasing real numbers.

The boundary of $[a, b]$ is the set of points $\{a, b\}$. To the interval $[a, b]$ in canonical orientation, we define the induced orientation on its boundary to be $\{(a, -), (b, +)\}$, and vice-versa for the interval $[a, b]_-$ in the reverse orientation. ◊

STEP 2. The next step is to define the integral of a one-form over an interval $[a, b]$ with a choice of orientation. This is what we did in Definition 3.1.1.

Definition 5.1.5 The integral of a one-form over an oriented interval $[a, b]_\pm$. Let ω be a one-form on $U \subseteq \mathbb{R}$, with $[a, b] \subset U$. We define **the integral of ω over the oriented interval** $[a, b]_\pm$ as:

$$\int_{[a,b]_\pm} \omega = \pm \int_a^b f(x)\, dx,$$

where on the right-hand-side we use the standard definition of definite integrals from calculus. ◊

STEP 3. Next step: introduce parametric curves $\alpha : [a, b] \to \mathbb{R}^n$. This was done in Definition 3.2.1. We do not repeat the definition here, but simply restate that it induces an orientation on the image curve $C = \alpha([a, b]) \subset \mathbb{R}^n$, and on its boundary $\partial C = \{(\alpha(a), -), (\alpha(b), +)\}$ if it is not closed. [1] We will write $\partial\alpha$ to denote the boundary of the parametric curve with its induced orientation.

STEP 4. We can then define the integral over a parametric curve α using the pullback. This is Definition 3.3.2.

Definition 5.1.6 (Oriented) line integrals. Let ω be a one-form on an open subset U of $\mathbb{R}^n$, and let $\alpha : [a,b] \to \mathbb{R}^n$ be a parametric curve whose image $C = \alpha([a,b]) \subset U$. We define the **integral of ω along α** as follows:

$$\int_\alpha \omega = \int_{[a,b]} \alpha^* \omega,$$

where the integral one the right-hand-side is defined in Definition 5.1.5. $\diamondsuit$

STEP 5. Then, in Lemma 3.3.5, we showed that the integral above is invariant under orientation-preserving reparametrizations, and changes sign under orientation-reversing reparametrizations. Thus we can think of the integral as being defined intrinsically in terms of the image curve $C = \alpha([a,b])$ and a choice of orientation.

STEP 6. Finally we consider the case of an exact one-form $\omega = df$. In this case, the integral simplifies drastically. Let us first look at what happens when we integrate an exact one-form over an interval $[a,b] \subset \mathbb{R}$.

Theorem 5.1.7 The Fundamental Theorem of Calculus. *Let f be a zero-form on $U \subseteq \mathbb{R}$, and $[a,b] \subset U$ with its canonical orientation. Let $\partial([a,b]) = \{(a,-),(b,+)\}$ be the boundary of the interval with its induced orientation. Then*

$$\int_{[a,b]} df = \int_{\partial([a,b])} f,$$

which is the Fundamental Theorem of Calculus (part II).

It may not seem obvious that this is the Fundamental Theorem of Calculus, but it is! On the left-hand-side, from our definition of integration, we have:

$$\int_{[a,b]} df = \int_a^b \frac{df}{dx}\, dx.$$

On the right-hand-side, from our definition of integration of zero-forms over oriented points (Definition 5.1.2), we get:

$$\int_{\partial([a,b])} f = f(b) - f(a).$$

So the theorem above is

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a),$$

which is the Fundamental Theorem of Calculus (part II). Cool!

Now let us move on to integration of exact one-forms over parametric curves.

[1] This is clear since the domain of a parametric curve is always the interval $[a,b]$ with its canonical orientation. Since the canonical orientation induces the orientation $\{(a,-),(b,+)\}$ on the boundary of the interval, it induces the orientation $\{(\alpha(a),-),(\alpha(b),+)\}$ on the boundary of the parametric curve.

Theorem 5.1.8 The Fundamental Theorem of line integrals. *Let f be a zero-form on $U \subseteq \mathbb{R}^n$, and $\alpha : [a,b] \to \mathbb{R}^n$ be a parametric curve whose image $C = \alpha([a,b]) \subset U$. The boundary of the parametric curve, with its induced orientation, consists in the two oriented points $\partial\alpha = \{(\alpha(a), -), (\alpha(b), +)\}$ if the image curve is not closed; otherwise it is the empty set. Then:*

$$\int_\alpha df = \int_{\partial\alpha} f.$$

Again, the statement above may look different from Theorem 3.4.1, but it is really the same thing. Indeed, the integral on the right-hand-side should be understood as the integral of a zero-form as in Definition 5.1.2. We can thus write:

$$\int_{\partial\alpha} f = \int_{\{(\alpha(a),-),(\alpha(b),+)\}} f = f(\alpha(b)) - f(\alpha(a)),$$

which is the statement in Theorem 3.4.1. In particular, if the curve is closed, $\partial\alpha$ is the empty set, and the right-hand-side vanishes, as in Corollary 3.4.3.

5.1.4 Exercises

1. Evaluate the integral of the zero-form $f(x,y,z) = \sin(x) + \cos(xyz) + e^{xy}$ at the set of oriented points $S = \{(p,+), (q,-), (r,+)\}$ with

$$p = (0,0,5), \qquad q = \left(\frac{\pi}{2}, \frac{2}{\pi}, \pi\right), \qquad r = (\pi, 1, 1).$$

Solution. By definition, the integral is

$$\begin{aligned}\int_S f =& f(p) - f(q) + f(r)\\ =& f(0,0,5) - f\left(\frac{\pi}{2}, \frac{2}{\pi}, \pi\right) + f(\pi,1,1)\\ =&(0+1+1) - (1-1+e) + (0-1+e^\pi)\\ =&1 + e^\pi - e.\end{aligned}$$

2. Let f be a zero-form on $\mathbb{R}^n$, and suppose that $a \in \mathbb{R}^n$ is a zero of the function f. Show that the integral of f at the point a does not depend on the orientation of the point.

Solution. Let $(a, \pm)$ be the point $a \in \mathbb{R}^n$ with the positive and negative orientations. The integral of f at $(a,\pm)$ is:

$$\begin{aligned}\int_{(a,\pm)} f =& \pm f(a)\\ =&0,\end{aligned}$$

since a is a zero of f. Since the result is the same regardless of what orientation the point a has, we conclude that the integral does not depend on the orientation of the point. (That's of course only because the point a is a zero of f, that wouldn't be true for an arbitrary point).

3. Suppose that f is a smooth function on $\mathbb{R}^n$, and let $p, q \in \mathbb{R}^n$ be two distinct points such that $f(p) = f(q)$. Show that the line integral of df along any curve starting at p and ending at q is zero.

Solution. Let α be any parametric curve whose image starts at p and ends at q. By the Fundamental Theorem of line integrals, we know that

$$\int_\alpha df = \int_{\partial\alpha} f = \int_{\{(p,-),(q,+)\}} f = f(q) - f(p) = 0,$$

where the last equality follows since we assume that $f(p) = f(q)$. Therefore, the line integral of df along any such parametric curve α is zero.

4. Let f be a zero-form on $\mathbb{R}^n$, and $S = \{(a,+),(-a,-)\}$ for some point $a \in \mathbb{R}^n$ ($-a$ denotes the point in $\mathbb{R}^n$ whose coordinates are minus those of a). Show that

$$\int_S f = \begin{cases} 0 & \text{if } f \text{ is even,} \\ 2f(a) & \text{if } f \text{ is odd.} \end{cases}$$

Solution. The integral of the zero-form is

$$\int_S f = \int_{\{(a,+),(-a,-)\}} f = f(a) - f(-a).$$

If f is even, $f(-a) = f(a)$, and hence

$$\int_S f = f(a) - f(-a) = f(a) - f(a) = 0.$$

If f is odd, $f(-a) = -f(a)$, and hence

$$\int_S f = f(a) - f(-a) = f(a) - (-f(a)) = 2f(a).$$

5. Let $\omega = 2xy\ dx + x^2\ dy$ be a one-form on $\mathbb{R}^2$. Suppose that

$$\int_\alpha \omega = 5$$

for some parametric curve $\alpha : [a,b] \to \mathbb{R}^2$. Show that the image curve $C = \alpha([a,b])$ is not a closed curve, i.e. it must have boundary points.

Solution. First, we notice that ω is exact. Indeed, let $f(x,y) = x^2y$. Then

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy = 2xy\ dx + x^2\ dy = \omega.$$

But then, by the Fundamental Theorem of line integrals, we know that

$$\int_\alpha \omega = \int_\alpha df = \int_{\partial\alpha} f.$$

In particular, if the image curve is closed, then the boundary set is empty, i.e. $\partial\alpha = \emptyset$, and the right-hand-side is zero. But the question states that it is non-zero; it is equal to 5. Therefore, the image curve cannot be closed.

6. You want to impress your calculus teacher, and you tell her that "integration by parts" can be rewritten as the "simple" statement that

$$\int_{[a,b]} d(fg) = \int_{\partial([a,b])} fg,$$

i.e. it is just the Fundamental Theorem of Calculus (part II) for a product of functions (f, g are differentiable functions on $\mathbb{R}$).

Explain why this is equivalent to integration by parts for definite integrals.

Solution. First, using the graded product rule for the exterior derivative, we know that $d(fg) = gdf + fdg$. So we can write the left-hand-side as

$$\int_{[a,b]} d(fg) = \int_{[a,b]} (gdf + fdg) = \int_a^b gdf + \int_a^b fdg.$$

As for the right-hand-side, the boundary of the interval is $\partial([a,b]) = \{(a,-),(b,+)\}$. So it can be rewritten as

$$\int_{\partial([a,b])} fg = f(b)g(b) - f(a)g(a).$$

Putting this together and rearranging a bit, we get

$$\int_a^b fdg = fg\Big|_a^b - \int_a^b gdf,$$

which is the statement of integration by parts for definite integrals.

5.2 Orientation of a region in $\mathbb{R}^2$

We proceed with building our theory of integration for two-forms. Step 1: we define the orientation of a closed bounded region in $\mathbb{R}^2$, the canonical orientation, and the induced orientation on its boundary.

Objectives

You should be able to:

- Relate the orientation of $\mathbb{R}^n$ to a choice of ordered basis.
- Determine whether two choices of ordered bases on $\mathbb{R}^n$ induce the same or opposite orientations.
- Define the orientation of a closed bounded region in $\mathbb{R}^2$.
- Determine the induced orientation on its boundary.

5.2.1 Orientation of $\mathbb{R}^n$

Recall that a choice of orientation on $\mathbb{R}$ is a choice of direction: either that of increasing real numbers, or of decreasing real numbers. We defined the direction of increasing real numbers as being the positive orientation, and called it the canonical orientation. We now generalize this to $\mathbb{R}^n$.

Definition 5.2.1 Orientation of $\mathbb{R}^n$. An **orientation** of the vector space $\mathbb{R}^n$ is determined by a choice of ordered basis on $\mathbb{R}^n$. We think of the orientation as a "twirl", which starts at the first basis vector, then rotates to the second basis vector, to the third, and so on and so forth. Ordered bases that generate the same twirl correspond to the same orientation of the vector space. ♢

What this definition is saying is that to give an orientation to $\mathbb{R}^n$, we pick a choice of ordered basis. But not all ordered bases give rise to different orientations; if they generate the same twirl, they correspond to the same orientation of the vector space. If you think about it carefully (think about $\mathbb{R}^2$ and $\mathbb{R}^3$ first), you will see that for any $\mathbb{R}^n$, there are only two distinct choices of twirl, and hence **only two distinct choices of orientation**.

This is due to a basic fact in linear algebra. Any two ordered bases of $\mathbb{R}^n$ are related by a linear transformation with non-vanishing determinant. One can see that two ordered bases that are related by a linear transformation with positive determinant generate the same twirl, while they generate opposite twirl if they are related by a linear transformation with negative determinant. So we can group oriented bases into two equivalence classes, depending on whether they are related by linear transformations with positive or negative determinants. These equivalence classes of oriented bases correspond to the two choices of orientation on $\mathbb{R}^n$.

Next we define the notion of canonical orientation.

Definition 5.2.2 Canonical orientation of $\mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ be coordinates on $\mathbb{R}^n$, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the ordered basis

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \qquad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \qquad \ldots, \qquad \mathbf{e}_n = (0, \ldots, 0, 1),$$

with $\mathbf{e}_i$ the unit vector pointing in the positive x_i-direction. We call the orientation described by this ordered basis the **canonical orientation** of $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$, and denote it by $(\mathbb{R}^n, +)$. We denote by $(\mathbb{R}^n, -)$ the vector space $\mathbb{R}^n$ with the opposite choice of orientation. ♢

As always, the definition will be clearer with low-dimensional examples. First, let us show that we recover our previous definition of orientation in $\mathbb{R}$.

Example 5.2.3 Orientation of $\mathbb{R}$ and choice of positive or negative direction. If x is a coordinate on $\mathbb{R}$, the canonical orientation is specified by the basis vector $\mathbf{e}_1 = 1$ in the positive x-direction. Thus the canonical orientation corresponds to the direction of increasing real numbers, as mentioned before. We denote it by $(\mathbb{R}, +)$.

Another choice of basis in $\mathbb{R}$ would be $\mathbf{f}_1 = -1$. It is related to $\mathbf{e}_1$ by the linear transformation $\mathbf{f}_1 = -\mathbf{e}_1$, which has negative determinant. Thus $\mathbf{f}_1$ generates the opposite orientation; indeed, it points in the direction of decreasing real numbers, which correspond to the orientation $(\mathbb{R}, -)$. So we recover our previous definition of orientation for $\mathbb{R}$. □

Example 5.2.4 Orientation of $\mathbb{R}^2$ and choice of counterclockwise or clockwise rotation. Let (x, y) be coordinates on $\mathbb{R}^2$. The canonical orientation is specified by the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, with $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ being the unit vectors pointing in the positive x- and y-directions. In two dimensions, the twirl generated by an ordered basis corresponds to a choice of direction of rotation, from the first basis vector to the second.

So one can think of an orientation of $\mathbb{R}^2$ as being a choice of direction of rotation. We see that the canonical basis corresponds to the choice of counterclockwise rotation. Thus $(\mathbb{R}^2, +)$ corresponds to $\mathbb{R}^2$ with the choice of counterclockwise rotation, which is the canonical orientation.

One could have instead chosen the ordered basis $\{\mathbf{f}_1, \mathbf{f}_2\} = \{\mathbf{e}_2, \mathbf{e}_1\}$ on $\mathbb{R}^2$ (the order of the vectors is important). Rotating from $\mathbf{e}_2$ to $\mathbf{e}_1$ corresponds to a clockwise rotation, so this ordered basis should induce the opposite orientation $(\mathbb{R}^2, -)$. Indeed, the two bases are related by the linear transformation (writing basis vectors as column vectors):

$$\mathbf{f}_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{e_i}, \qquad \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

They therefore generate opposite orientations, as expected. □

Example 5.2.5 Orientation of $\mathbb{R}^3$ and choice of right-handed or left-handed twirl. Let (x, y, z) be coordinates on $\mathbb{R}^3$. The canonical orientation is specified by the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ being the unit vectors pointing in the positive x-, y- and z-directions. In three dimensions, the twirl generated by an ordered basis corresponds to a choice of left-handed or right-handed orientation. We can think of the first rotation from the first basis vector to the second as being represented by curling your fingers, and then the rotation from the second basis vector to the third as being in the direction of your thumb. The canonical basis thus corresponds to the choice of right-handed twirl, which is the canonical orientation on $\mathbb{R}^3$ and denoted by $(\mathbb{R}^3, +)$.

Another ordered basis on $\mathbb{R}^3$ could have been $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$. Following the twirl with your fingers and thumb, you will see that this corresponds to a left-handed twirl. So we expect it to correspond to the opposite orientation $(\mathbb{R}^3, -)$. Indeed, it is related to the canonical basis by the linear transformation (writing basis vectors as column vectors):

$$\mathbf{f}_i = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{e}_i, \qquad \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

The two ordered bases thus generate opposite orientations, as expected. □

5.2.2 Orientation of a closed bounded region in $\mathbb{R}^2$

Let us now focus on $\mathbb{R}^2$. Now that we defined the notion of orientation for the vector space $\mathbb{R}^2$, we can define the orientation of a closed bounded region in $\mathbb{R}^2$, just as we did for closed intervals in $\mathbb{R}$. But let us first define more carefully what we mean by a closed bounded region in $\mathbb{R}^2$. We use this to also recall the definition of path connectedness and simple connectedness (see Subsection 3.6.2), which will be useful later.

Definition 5.2.6 Regions in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a region in $\mathbb{R}^2$.

- A **boundary point** is a point $p \in \mathbb{R}^2$ such that all disks centered at p contain points in D and also points not in D. We define the **boundary** of D, which we denote by ∂D, to be the set of all boundary points of the region D.

- We say that a region D is **closed** if it contains all its boundary points, that is, $\partial D \subseteq D$. We say that it is **open** if it contains none of its boundary points, that is, $D \cap \partial D = \emptyset$.
- We say that a region D is **bounded** if it is contained within a finite disk. In other words, it is bounded if it is finite in extent.
- We say that a region D is **path connected** (or **connected**) if any two points in D can be connected by a path within D. In other words, it is path connected if it has only one component.
- We say that a region D is **simply connected** if it is path connected and all simple closed curves (loops) in D can be continuously contracted to a point within D. In other words, it is simply connected if it has only one component and no holes.

$\Diamond$

Remark 5.2.7 We will often consider closed, simply connected, bounded regions $D \subset \mathbb{R}^2$. Such a region can be constructed by considering a simple closed curve $C \subset \mathbb{R}^2$, and letting the region D be the closed curve and its interior. The boundary of the region is then $\partial D = C$, i.e. the closed curve that we started with. With this construction, it is clear that D is bounded, and it is simply connected since it has one component and no holes.

We note however that the closed curve C does not have to be a parametric curve; it may have kinks and corners, that is, it could be piecewise parametric. For instance, C could be a triangle, or a rectangle, etc. It needs to be simple however, i.e. not have self-intersection.

Next we define the orientation of a closed bounded region $D \subset \mathbb{R}^2$. Recall that we defined the orientation of a closed interval in $\mathbb{R}$ as being a choice of direction, just as for $\mathbb{R}$; so we can think of the orientation of an interval as being induced by a choice of orientation on the surrounding vector space $\mathbb{R}$. We do the same for regions in $\mathbb{R}^2$.

Definition 5.2.8 Orientation of a closed bounded region in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a closed bounded region in $\mathbb{R}^2$, and choose an orientation on $\mathbb{R}^2$. We define the **orientation of the region** D as being the orientation induced by the surrounding vector space $\mathbb{R}^2$. We write D_+ for the region $D \subset \mathbb{R}^2$ with the canonical (counterclockwise) orientation on $\mathbb{R}^2$, and D_- for the region with the opposite (clockwise) orientation. When we write D without specifying the orientation, we always mean the region D with its canonical orientation. $\Diamond$

As a last step, given a closed bounded region $D \subset \mathbb{R}^2$ with a choice of orientation, we need to define the induced orientation on its boundary ∂D. In the one-dimensional case, given a closed interval $[a, b] \in \mathbb{R}$ with canonical orientation, we defined the induced orientation on its boundary to be $\{(a, -), (b, +)\}$. We do something similar in two dimensions. In this case, the boundary ∂D is a curve (which may have more than one components), and hence the induced orientation should be a choice of direction on each component.

Definition 5.2.9 Induced orientation on the boundary of a region in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a closed bounded region with canonical orientation, and ∂D its boundary. Imagine that the region D is on the floor and that you are walking on its boundary. We define the induced orientation on each boundary component as being the direction of travel along the boundary keeping the region on your left. If D has the opposite orientation, the induced orientation on the boundary is the opposite direction of travel. $\Diamond$

This will be clearer with some examples.

Example 5.2.10 Closed disk in $\mathbb{R}^2$. Consider the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\},$$

which is a disk of radius one centered at the origin. The boundary points in D are the points on the circle $x^2 + y^2 = 1$. Thus ∂D is the circle of radius one centered at the origin. Since it is included in D, this means that D is closed. It is clearly bounded, as it can be contained within a disk. It is also simply connected.

We can give D the canonical orientation induced by the standard choice of ordered basis on $\mathbb{R}^2$, which is the choice of counterclockwise direction of rotation. The induced orientation on the boundary is the counterclockwise direction of motion along the circle, as this is the direction that one needs to move along the circle to keep its interior on the left. □

Example 5.2.11 Closed square in $\mathbb{R}^2$. Consider the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

This corresponds to a square and its interior centered at the origin and with side length 2. It is bounded as it can be contained within a finite disk. Its boundary ∂D is the square itself. As it is included in D, D is closed. It is also simply connected. If we choose the canonical orientation on D, the induced orientation on ∂D corresponds to moving counterclockwise along the square. □

Example 5.2.12 Annulus in $\mathbb{R}^2$. Consider the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\}.$$

This corresponds to an annulus with inner radius 1 and outer radius 2. D is certainly bounded as it is contained within a finite disk. Its boundary has two components, $\partial D = \partial D_1 \cup \partial D_2$, where ∂D_1 is the inner circle of radius 1 and ∂D_2 is the outer circle of radius 2. While the region is connected, it is not simply connected, as there is a hole in the middle.

Suppose that we choose the canonical orientation on D. What is the induced orientation on the boundary? We have to look at the two components separately. First, along the outer radius ∂D_2, we need to move in the counterclockwise direction to keep the interior of the annulus on the left. Thus the induced orientation is counterclockwise. However, for the inner circle ∂D_1, we need to move in the clockwise direction to keep the interior of the annulus on the left. So the induced orientation on ∂D_1 is clockwise. □

Remark 5.2.13 There is another way that one can think of the induced orientation on the boundary of a region in $\mathbb{R}^2$. It is a little bit more subtle, and while it is not needed at this stage, it will be useful to generalize to regions in $\mathbb{R}^3$ and $\mathbb{R}^n$, so let us mention it here.

Consider first the case where $D \subset \mathbb{R}^2$ is a closed, simply connected, bounded region. Then ∂D is a simple closed curve. Since a curve is a one-dimensional subspace of $\mathbb{R}^2$, at a point $p \in \partial D$ there are two choices of (length one) normal vectors: one that points "inwards" (towards D), and one that points "outwards" (away from D). For all points $p \in \partial D$, we pick the normal vector that points outwards. This defines an orientation on ∂D as follows; start with the normal vector, and rotate to a tangent vector in a way that reproduces the orientation (the twirl) on the ambient space $\mathbb{R}^2$. This defines a choice of tangent vector at all points on ∂D, which defines the induced direction of motion or orientation on the curve. It is easy to see in a figure that if the ambient space $\mathbb{R}^2$ is equipped with canonical orientation, choosing the normal vectors pointing outwards corresponds to choosing the direction of motion keeping the region on the left.

If the region is not simply connected, its boundary may have many components. Do the same construction for each component, always choosing the normal vectors pointing outwards (away from D). This will induce the orientation on each component corresponding to the direction of motion keeping the region on the left.

This construction in terms of normal vectors is more subtle, but it directly generalizes to closed bounded regions in $\mathbb{R}^n$, which is nice.

5.2.3 Exercises

1. Determine whether the following regions are bounded, closed, connected, and/or simply-connected.

(a) $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y \geq 3\}$.

(b) $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

(c) $D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1] \cup [3, 4], y \in [0, 1]\}$.

(d) $D = \{(x, y) \in \mathbb{R}^2 \mid \dfrac{x^2}{4} + \dfrac{y^2}{9} \leq 1\}$.

Solution.

(a) D extends forever in the positive x and y directions, so it is not bounded (it is not of finite extent). The boundary points are the points in D with $x = 2$ or $y = 3$, since they are at the edge of the region. All those points are included in D, so D is closed. It is connected, as it has only one component, and it is simply-connected, as there is no hole.

(b) D is the unit disk of radius one, without its boundary the circle of radius one. It is bounded, since it can be contained within a disk (it is of finite extent). It is not closed, since the boundary of D is the circle of radius one, which is not included in D. It is connected (one component) and simply-connected (no hole).

(c) D consists in two separate square components. It is bounded, since the two square components can be contained within a disk (finite extent). It is closed, since the boundary points are the edges of the squares, which are all included in D. It is however not connected (two components), and hence also not simply-connected.

(d) D is the region bounded by ellipse centered at the origin. It is bounded (finite extent), closed (the ellipse itself, which is the boundary, is included in D), connected (one component), and simply-connected (no hole).

2. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis on $\mathbb{R}^3$. Show that the ordered basis $\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\}$ induces the same orientation on $\mathbb{R}^3$ as the canonical basis.

Solution. To show that two ordered bases induce the same orientation, we need to show that they are related by a linear transformation with positive determinant. If we write $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\}$ for the ordered basis, we see that it is related to the canonical basis by the linear transformation (thinking of the basis vectors as column vectors):

$$\mathbf{f}_i = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{e}_i.$$

Indeed,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

But

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 1,$$

and hence the two ordered bases induce the same orientation on $\mathbb{R}^3$.

3. Let D be the upper half of a disk of radius one, including its boundary. Suppose that D is given the canonical orientation. Write its boundary, with the induced orientation, as an oriented parametric curve.

Solution. The upper half of a disk of radius one is the region of $\mathbb{R}^2$ defined by:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y \geq 0\}.$$

Its boundary ∂D is the upper half of the circle, and the x-axis between $x = -1$ and $x = 1$. Since D has canonical orientation, the induced orientation corresponds to walking along the boundary curve keeping the region to the left, which means going counterclockwise along the boundary.

To realize ∂D as a parametric curve, we need to split it in two, since there are corners where the upper half disk meets the x-axis. Let C be the upper half disk, and L be the part of the x-axis in the boundary. We can parametrize the two curves separately. For C we take $\alpha_1 : [0, \pi] \to \mathbb{R}^2$ with

$$\alpha_1(t) = (\cos(t), \sin(t)).$$

This has the correct orientation, as it goes counterclockwise around the circle. As for the L, we need to parametrize the line $y = 0$ from $x = -1$ to $x = 1$. We take $\alpha_2 : [-1, 1] \to \mathbb{R}^2$

with

$$\alpha_2(t) = (t, 0).$$

4. Let A be a closed disk of radius four centered at the origin, and B be the region

$$B = \{(x, y) \in \mathbb{R}^2 \mid x \in (-1, 1), y \in (-1, 1)\}.$$

Let D be the region that consists of all points in A that are not in B. Is D bounded? Closed? Connected? Simply-connected? What is the boundary of D? And if D is given the canonical orientation, what is the induced orientation on the boundary ∂D?

Solution. We note that B is the interior of a square of side length two centered at the origin. It is enclosed within A, which is a disk of radius four. Therefore, D is certainly bounded, since it is of finite extent.

The boundary points of D consists of all points on its outer boundary, which is the circle of radius four, and on its inner boundary, which consists on the points on the square centered at the origin. Thus, it has two separate components. But all the boundary points are included in D, and hence D is closed.

It is connected, as it has only one component. However, it is not simply connected, as any closed loop going around the inner missing square cannot be contracted to a point within D.

What is the induced orientation on the boundary? Let us denote by C_1 the outer boundary consists of the circle of radius four, and C_2 the inner boundary consisting of the square of side length two. Since D has canonical orientation, the induced orientation on the boundary will be obtained by walking along the boundary components keeping the region to the left. Along the outer boundary C_1, we will keep the region to the left if we walk counterclockwise. However, along the inner boundary C_2, we will keep the region to the left if we walk clockwise. Therefore, the induced orientation is councterclockwise on C_1 and clockwise on C_2.

5. The parametric curve $\alpha : [0, \pi] \to \mathbb{R}^2$ with

$$\alpha(t) = (\sin(t), \sin(2t))$$

is shown in the following figure:

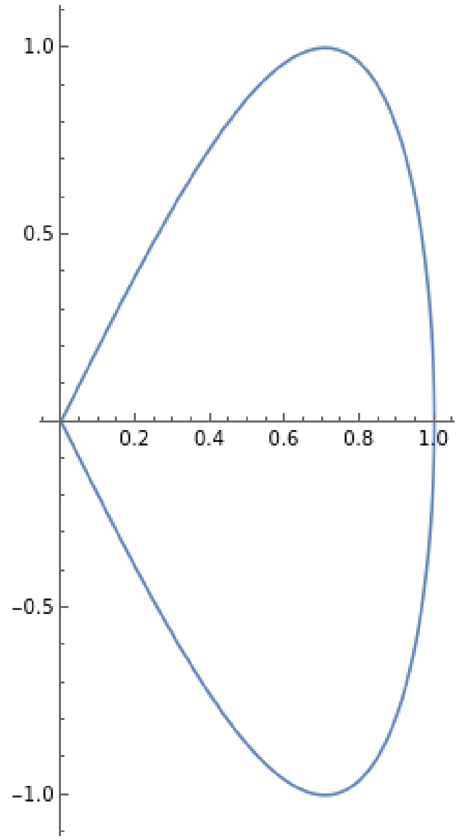


Figure 5.2.14 The parametric curve $\alpha : [0, \pi] \to \mathbb{R}^2$ with $\alpha(t) = (\sin(t), \sin(2t))$.

Suppose that D is the region consisting of the curve and its interior. What should the orientation of the region D be so that the induced orientation on its boundary is the same as the orientation of the parametric curve?

Solution. Let us first find the orientation of the parametric curve. The tangent vector is

$$\mathbf{T}(t) = (\cos(t), 2\cos(2t)).$$

In particular, at the origin, the tangent vector is $\mathbf{T}(0) = (1, 2)$. It points upwards and in the positive x-direction. Therefore, we see that the parametric curve has clockwise orientation.

The region D is the region consisting of the curve and its interior. If we walk clockwise along the curve, the region is on our right. This means that we must give D a clockwise (or negative) orientation if we want the induced orientation on the boundary to be the same as the orientation of the parametric curve.

5.3 Integrating a two-form over a region in $\mathbb{R}^2$

Step 2: we define the integral of a two-form on $U \subset \mathbb{R}^2$ along a closed bounded domain in U with a choice of orientation. The definition is in terms of standard double integrals from calculus. We show that the resulting integral is invariant under orientation-preserving reparametrizations, and changes sign under orientation-reversing reparametrizations. This is closely connected to the transformation formula for double integrals.

Objectives

You should be able to:

- Define the integral of a two-form over a closed bounded region in $\mathbb{R}^2$.
- Show that invariance under orientation-preserving reparametrizations recovers the transformation formula for double integrals.

5.3.1 The integral of a two-form over an oriented closed bounded region in $\mathbb{R}^2$

We define the integral of a two-form over an oriented closed bounded region in $\mathbb{R}^2$.

Definition 5.3.1 Integral of a two-form over an oriented closed bounded region in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a closed bounded region. Let (x, y) be coordinates on $\mathbb{R}^2$. The canonical $(+)$ orientation on D is described by the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$, with $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ the unit vectors pointing in the positive x- and y-directions. Let $\omega = f\ dx \wedge dy$ be a two-form on an open subset $U \subseteq \mathbb{R}^2$ such that $D \subset U$. Then, we define the **integral of ω over D with canonical orientation** as:

$$\int_{D_+} \omega = \iint_D f\ dA,$$

where on the right-hand-side we mean the standard double integral from calculus of the function f over the region D. If D is given the opposite orientation, we define the integral of ω over D_- as:

$$\int_{D_-} \omega = -\iint_D f\ dA.$$

$\Diamond$

Remark 5.3.2 A bit of notation: we will always use double or triple integral signs to denote the standard double and triple integrals from calculus, while we will use only one integral sign when we are integrating a differential form.

It is probably worth recalling here how double integrals are defined, from your previous calculus course. If D is a **rectangular domain**, that is,

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], y \in [c, d]\},$$

then the double integral is defined as an iterated integral:

$$\iint_D f\ dA = \int_c^d \int_a^b f\ dxdy.$$

The notation here means that the inner integral is an integral with respect to x (while keeping

y fixed), while the outer integral is with respect to y.[1] We recall Fubini's theorem, which states that the order of integration does not matter:

$$\int_c^d \int_a^b f \ dxdy = \int_a^b \int_c^d f \ dydx,$$

that is, it does not matter whether you integrate in x first and then in y or the other way around.

To integrate over a more general closed bounded region D, we proceed as follows. Since D is bounded, we can take it to be inside a rectangular region. We can then extend the function f to the rectangular region by setting it to zero everywhere outside D. The double integral over D is then defined to be the integral of the extended function over the rectangular region, which can be written as an iterated itegral as above.

There are two types of regions that give rise to nice iterated integrals. If D can be written as follows:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], u(x) \leq y \leq v(x)\},$$

with $u, v : \mathbb{R} \to \mathbb{R}$ continuous functions, we say that the region is **x-supported** (or of **type I**). In this case, one can show that the double integral can be written as the following iterated integral:

$$\iint_D f \ dA = \int_a^b \int_{u(x)}^{v(x)} f \ dydx,$$

where the inner integral is with respect to x (keeping y fixed), while the outer integral is with respect to y.

If instead D can be written as:

$$D = \{(x,y) \in \mathbb{R}^2 \mid y \in [c,d], u(y) \leq x \leq v(y)\},$$

we say that D is **y-supported** (or of **type II**). The double integral is then the iterated integral:

$$\iint_D f \ dA = \int_c^d \int_{u(y)}^{v(y)} f \ dxdy,$$

with the inner integral being with respect to y (keeping x fixed), and the outer integral with respect to y.

Note that rectangular regions are particular cases of both x-supported and y-supported regions. If a region D is either x-supported or y-supported, we say that it is **recursively supported**. Most of the regions that we will deal with will be either recursively supported regions, or regions that can be expressed as unions of recursively supported regions.

Example 5.3.3 Integral of a two-form over a rectangular region with canonical orientation. Consider the two-form $\omega = xy \ dx \wedge dy$ on $\mathbb{R}^2$, and the closed bounded region

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, -1 \leq y \leq 3\},$$

[1] Note that on the right-hand-side here there is no wedge product: this is not the integral of a two-form, it is an interated integral in x and y as you have seen in calculus. The inner integral is with respect to x (keeping y fixed), while the outer integral is with respect to y.

equipped with the canonical orientation. Then

$$\begin{aligned}\int_{D_+} \omega =& \iint_D xy\ dA \\ =& \int_{-1}^{3}\int_0^2 xy\ dxdy \\ =& \int_{-1}^{3} y \left[\frac{x^2}{2}\right]_{x=0}^{x=2} dy \\ =& 2\int_{-1}^{3} y\ dy \\ =& y^2\Big|_{-1}^{3} \\ =& 8.\end{aligned}$$

To evaluate the double integral, we used the standard procedure for evaluating double integrals over rectangular regions as iterated integrals, with the inner integral being with respect to x (keeping y constant), and the outer integral being with respect to y. □

Example 5.3.4 Integral of a two-form over an x-supported (or type I) region with canonical orientation. Consider the two-form $\omega = xe^y\ dx \wedge dy$, and the region

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [1,2], \ln(x) \leq y \leq \ln(2x)\}.$$

D is an x-supported region. The integral then reads:

$$\begin{aligned}\int_{D_+} \omega =& \int_1^2 \int_{\ln(x)}^{\ln(2x)} xe^y\ dydx \\ =& \int_1^2 x\,[e^y]_{\ln(x)}^{\ln(2x)}\ dx \\ =& \int_1^2 \left(xe^{\ln(2x)} - xe^{\ln(x)}\right) dx \\ =& \int_1^2 \left(2x^2 - x^2\right) dx \\ =& \int_1^2 x^2\ dx \\ =& \frac{8}{3} - \frac{1}{3} \\ =& \frac{7}{3}.\end{aligned}$$

□

Remark 5.3.5 It is worth pointing out here that Definition 5.3.1 is actually quite subtle. We defined the integral of the two-form $\omega = f\ dx \wedge dy$ over D_+ as

$$\int_{D_+} f\ dx \wedge dy = \iint_D f\ dA,$$

where the right-hand-side is the standard double integral of a function in calculus. The subtelty is that the right-hand-side is not an oriented integral, while the left-hand-side is. Indeed, suppose for simplicity that $D = [a, b] \times [c, d]$ is a rectangular region, as in Example 5.3.3. Then, we can interpret the right-hand-side as an iterated integral:

$$\iint_D f \ dA = \int_c^d \int_a^b f \ dxdy.$$

But then, by Fubini's theorem, we know that we can exchange the order of integration without issue. That is,

$$\iint_D f \ dA = \int_c^d \int_a^b f \ dxdy = \int_a^b \int_c^d f \ dydx.$$

At first sight, this may appear problematic, as one could be tempted to reinterpret the last integral as

$$\int_{D_+} f \ dy \wedge dx,$$

but since $dx \wedge dy = -dy \wedge dx$, this would be minus the integral we started with! But this is incorrect. The subtlety is in the choice of orientation.

The key is that in Definition 5.3.1, we started by choosing coordinates (x, y) on $\mathbb{R}^2$, and then we wrote the one-form $\omega = f \ dx \wedge dy$ using the basic two-form $dx \wedge dy$ in which the differentials dx and dy appear **in the same order** as the coordinates (x, y). This is important, as $dy \wedge dx = -dx \wedge dy$. So while we can exchange the order of the iterated integrals once we have written everything in terms of double integrals, when we write the integral in terms of differential forms, we must use the correct choice of basic two-form $dx \wedge dy$ with the differentials appearing in the same order as the coordinates of $\mathbb{R}^2$. That's because integrals of two-forms are oriented, while double integrals of functions are not.

Note that there is nothing special about the variables x and y. We could name the coordinates of $\mathbb{R}^2$ anything. For instance, if we choose coordinates (u, v) on $\mathbb{R}^2$, then the integral of a two-form $\omega = f \ du \wedge dv$ over a region D is equal to the double integral of f over D (with a positive sign in front) when D is endowed with the canonical orientation described by the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, with the basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ pointing in the positive directions of the coordinates u and v respectively.[2]

5.3.2 Integrals of two-forms over regions in $\mathbb{R}^2$ are oriented and reparametrization-invariant

We already mentioned that integrals of two-forms oriented. Let us now be a little more precise, and show that, with our definition, integrals of two-forms are invariant under orientation-preserving reparametrizations, and change sign under orientation-reversing reparametrizations.

Let us first define what we mean by orientation-preserving and orientation-reversing reparametrizations. We state the definition in $\mathbb{R}^2$, but it naturally generalizes to $\mathbb{R}^n$.

[2] In a more careful treatment, we would allow to write the two-form ω in terms of any basic two-form α. We would then define the orientation that is compatible with the basic two-form as corresponding to choices of oriented bases $\{\mathbf{u}, \mathbf{v}\}$ such that $\alpha(\mathbf{u}, \mathbf{v}) > 0$. Our definition would then state that the integral of a two-form written in terms of this basic two-form over a region D with the compatible orientation would be given by the double integral of the function over that region, while the integral over the region with opposite orientation would be given by minus the double integral.

Definition 5.3.6 Orientation-preserving reparametrizations of regions in $\mathbb{R}^2$. Let $D_1, D_2 \subset \mathbb{R}^2$ be recursively supported regions,[3] and $\phi : D_2 \to D_1$ a bijective function that can be extended to a C^1-function on an open subset $U \subseteq \mathbb{R}^2$ that contains D_2. We assume that ϕ is invertible (except possibly on the boundary of D_2), i.e. the determinant of its Jacobian $\det J_\phi$ is non-zero on the interior of D_2 (recall the definition of the Jacobian in Definition 4.7.5). We say that ϕ is an **orientation-preserving reparametrization** if $\det J_\phi > 0$ for all points in the interior of D_2, and that it is an **orientation-reserving reparametrization** if $\det J_\phi < 0$ for all points in the interiori of D_2. $\Diamond$

With this definition, we can now show that our theory of integration for two-forms over regions in $\mathbb{R}^2$ is oriented and reparametrization-invariant, our two guiding principles.

Lemma 5.3.7 Integrals of two-forms over regions in $\mathbb{R}^2$ are invariant under orientation-preserving reparametrizations. *Let $D_1, D_2 \subset \mathbb{R}^2$ be recursively supported regions, and ϕ a reparametrization as in Definition 5.3.6. Let ω be a two-form on an open subset $U \subseteq \mathbb{R}^2$ that contains D_1.*

- *If ϕ is orientation-preserving, then*
$$\int_{D_2} \phi^* \omega = \int_{D_1} \omega.$$
- *If ϕ is orientation-reversing, then*
$$\int_{D_2} \phi^* \omega = -\int_{D_1} \omega.$$

In other words, the integral is invariant under orientation-preserving reparametrizations, and changes sign under orientation-reversing reparametrizations.

Proof. The key is to use Lemma 4.7.7 to calculate $\phi^*\omega$. Since we are pulling back a two-form ω on an open subset $U \subseteq \mathbb{R}^2$, and that $\phi : V \to U$ with $V \subseteq \mathbb{R}^2$, we can apply the result of Lemma 4.7.7. We write $\omega = f\ dx \wedge dy$, and $\phi(u, v) = (x(u, v), y(u, v))$. The jacobian of the transformation is
$$J_\phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$
Then Lemma 4.7.7 tells us that
$$\phi^*\omega = f(\phi(u, v))(\det J_\phi)\ du \wedge dv.$$
This means that
$$\int_{D_2} \phi^*\omega = \int_{D_2} f(\phi(u, v))(\det J_\phi)\ du \wedge dv.$$
Using our definition of integration in Definition 5.3.1, we know that
$$\int_{D_1} \omega = \iint_{D_1} f(x, y)\ dxdy, \qquad \int_{D_2} \phi^*\omega = \iint_{D_2} f(\phi(u, v))(\det J_\phi)\ dudv,$$

[3] We could extend the statement more generally to closed bounded regions by taking unions of recursively supported regions.

where on the right-hand-side of each equation we mean the double integral for the recursively supported regions D_1 in the xy-plane and D_2 in the uv-plane.

But recall from your previous calculus course that double integrals satisfy a "transformation formula", or "change of variables formula", which is the natural generalization of the substitution formula for definite integrals. The transformation formula states that

$$\iint_{D_1} f(x,y)\ dxdy = \iint_{D_2} f(\phi(u,v))|\det J_\phi| dudv.$$

Note that there is now an absolute value around the determinant of the Jacobian. Thus, what this means is that if our transformation is such that $\det J_\phi > 0$, then $|\det J_\phi| = \det J_\phi$, and

$$\int_{D_1} \omega = \iint_{D_1} f(x,y)\ dxdy = \int_{D_2} f(\phi(u,v))(\det J_\phi)\ du \wedge dv = \int_{D_2} \phi^*\omega,$$

while if $\det J_\phi < 0$, then $|\det J_\phi| = -\det J_\phi$, and

$$\int_{D_1} \omega = \iint_{D_1} f(x,y)\ dxdy = -\int_{D_2} f(\phi(u,v))(\det J_\phi)\ du \wedge dv = -\int_{D_2} \phi^*\omega.$$

This is the statement of the lemma: integrals of two-forms are oriented and reparametrization-invariant! ■

What is particularly nice with the proof of the lemma is that **the transformation (or change of variables) formula for double integrals is simply the statement that integrals of two-forms over regions in $\mathbb{R}^2$ are invariant under orientation-preserving reparametrizations**! Isn't that cool? It explains why the determinant of the Jacobian appears; it comes from pulling back the two-form under the change of variables.

The fact that double integrals involve the absolute value of the determinant of the Jacobian, while our integrals do not (and change signs under orientation-reversing reparametrizations), is also interesting. As alluded to above, the reason is that our integrals are oriented, while standard double integrals in calculus are not.

Example 5.3.8 Area of a disk. Consider the basic two-form $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Let us define the following x-supported domain:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,1], -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}.$$

It is easy to see that D is a closed disk of radius one centered at the origin. The integral of the basic two-form ω over D should give us the area of the disk, namely π. We calculate:

$$\begin{aligned}\int_D \omega =& \iint_D\ dA\\ =& \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dydx\\ =& \int_{-1}^{1} \left(\sqrt{1-x^2} + \sqrt{1-x^2}\right)\ dx\\ =& 2\int_{-1}^{1} \sqrt{1-x^2}\ dx.\end{aligned}$$

To evaluate this integral, we do a trigonometric substitution, $x = \sin(\theta)$:

$$\begin{aligned}
2\int_{-1}^{1} \sqrt{1-x^2}\ dx =& 2\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2(\theta)}\cos(\theta)\ d\theta\\
=&2\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\ d\theta\\
=&\int_{-\pi/2}^{\pi/2} (1+\cos(2\theta))\ d\theta\\
=&\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+\frac{1}{2}\left(\sin(\pi)-\sin(-\pi)\right)\\
=&\pi.
\end{aligned}$$

We could have instead use a change of variables to evaluate this integral: polar coordinates. Define the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\phi(r,\theta) = (r\cos(\theta), r\sin(\theta)).$$

Then, if we define the region

$$D_2 = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,1], \theta \in [0,2\pi]\},$$

the map $\phi : D_2 \to D$ is bijective and invertible in the interior of D_2. The determinant of the Jacobian is

$$\det J_\phi = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r,$$

which is positive for all points in the interior of D_2. Thus ϕ is an orientation-preserving reparametrization, so the integral of the pullback $\phi^*\omega = r\ dr \wedge d\theta$ over D_2 should give us π again. Indeed, we get:

$$\begin{aligned}
\int_{D_2} \phi^*\omega =& \int_0^{2\pi}\int_0^1 (\det J_\phi)\ dr d\theta\\
=&\int_0^{2\pi}\int_0^1 r\ dr d\theta\\
=&\frac{1}{2}\int_0^{2\pi} d\theta\\
=&\pi.
\end{aligned}$$

Notice how easier the integral was! That's of course because polar coordinates are well suited for evaluating integrals over regions that have circular symmetry. □

5.3.3 Exercises

1. Evaluate the integral of the two-form $\omega = \frac{1}{(1+x+y)^2} dx \wedge dy$ over the rectangular region

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [1,2], y \in [2,3]\}$$

with canonical orientation.

Solution. By definition of the integral, we have:

$$\begin{aligned}
\int_D \omega &= \int_D \frac{1}{(1+x+y)^2}\ dx \wedge dy \\
&= \int_2^3 \int_1^2 \frac{1}{(1+x+y)^2}\ dxdy \\
&= \int_2^3 \left[-\frac{1}{1+x+y}\right]_{x=1}^{x=2}\ dy \\
&= \int_2^3 \left(\frac{1}{2+y} - \frac{1}{3+y}\right)\ dy \\
&= [\ln(2+y) - \ln(3+y)]_{y=2}^{y=3} \\
&= \ln(5) - \ln(6) - \ln(4) + \ln(5) \\
&= \ln\left(\frac{25}{24}\right).
\end{aligned}$$

2. Evaluate the integral of the two-form $\omega = \frac{\ln(y)}{xy}\ dx \wedge dy$ over the rectangular region

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [1,4], y \in [1,5]\}$$

with clockwise orientation.

Solution. By definition of the integral, we have (we add a minus sign since we are evaluating the integral with clockwise orientation):

$$\begin{aligned}
\int_D \omega &= -\int_D \frac{\ln(y)}{xy}\ dx \wedge dy \\
&= -\int_1^5 \int_1^4 \frac{\ln(y)}{xy}\ dxdy \\
&= -\int_1^5 \frac{\ln(y)}{y} [\ln(x)]_{x=1}^{x=4}\ dy \\
&= -\ln(4) \int_1^5 \frac{\ln(y)}{y}\ dy.
\end{aligned}$$

To evaluate the remaining definite integral, we do a substitution $u = \ln(y)$, $du = \frac{1}{y}\ dy$, and $y = 1$ goes to $u = \ln(1) = 0$, while $y = 5$ goes to $u = \ln(5)$. We get:

$$\begin{aligned}
\int_D \omega &= -\ln(4) \int_0^{\ln(5)} u\ du \\
&= -\ln(4) \left[\frac{u^2}{2}\right]_{u=0}^{u=\ln(5)} \\
&= -\frac{\ln(4)(\ln(5))^2}{2}.
\end{aligned}$$

3. Evaluate the integral of the two-form $\omega = \cos(y^3)\ dx \wedge dy$ over the region D bounded by the parabola $x = y^2$ and the lines $x = 0$, $y = 0$ and $y = 1$, with canonical orientation.

Solution. We first describe the region explicitly. It is shown in the figure below:

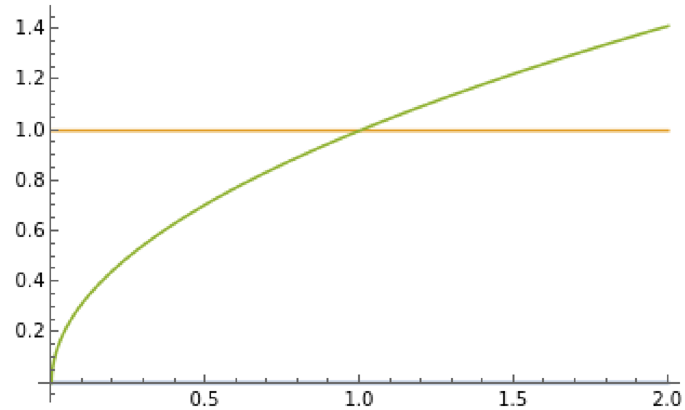


Figure 5.3.9 The region D is the region bounded by the two curves shown above, the y-axis, and the x-axis. The green curve is the upper half of the curve $x = y^2$, while the orange line is the line $y = 1$.

We can describe this region as an x-supported or y-supported region. If we write it as a y-supported region, we would write $0 \leq y \leq 1$, and $0 \leq x \leq y^2$. Thus

$$D = \{(x, y) \in \mathbb{R}^2 \mid y \in [0, 1], 0 \leq x \leq y^2\}.$$

If we write it as an x-supported region, we would write $0 \leq x \leq 1$, and $\sqrt{x} \leq y \leq 1$, that is,

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1], \sqrt{x} \leq y \leq 1\}.$$

Which of the two descriptions should we choose? If we use the second description (x-supported), we first need to integrate in y. But the function that we have to integrate is $\cos(y^3)$, which has no elementary antiderivative. So this is problematic. We will not run into this problem if we use the first description (y-supported); with this description, we first integrate in x, which is fine. And the next integral in y will then be easy, as we will see.

So we use the first description of the region as x-supported. The integral then becomes

$$\begin{aligned}\int_D \omega &= \int_D \cos(y^3)\ dx \wedge dy \\ &= \int_0^1 \int_0^{y^2} \cos(y^3)\ dxdy \\ &= \int_0^1 \cos(y^3)\, [x]_{x=0}^{x=y^2}\ dy \\ &= \int_0^1 y^2 \cos(y^3)\ dy.\end{aligned}$$

To evaluate the remaining definite integral, we do the substitution $u = y^3$, $du = 3y^2\ dy$. Then $y = 0$ becomes $u = 0$, and $y = 1$ becomes $u = 1$. We get:

$$\begin{aligned}\int_D \omega =& \frac{1}{3} \int_0^1 \cos(u)\ du \\ =& \frac{1}{3} \sin(u)\Big|_{u=0}^{u=1}\end{aligned}$$

$$=\frac{1}{3}\sin(1).$$

4. Consider the two-form $\omega = e^{(x^2+y^2)^2}\ dx \wedge dy$, and let D be the disk of radius one with counterclockwise orientation. Show that

$$\int_D \omega = 2\pi \int_0^1 re^{r^4}\ dr.$$

Solution. To show this, we change coordinates from Cartesian coordinates to polar coordinates. We define the function $\phi : D_2 \to D$, with

$$\phi(r, \theta) = (r\cos(\theta), r\sin(\theta)),$$

and

$$D_2 = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 1], \theta \in [0, 2\pi]\}.$$

It maps the rectangular region D_2 to the unit disk D. The determinant of the Jacobian of ϕ is

$$\begin{aligned}\det J_\phi =& \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ =& \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \\ =& r.\end{aligned}$$

As $r \in [0, 1]$, this is positive on the interior of D_2, and thus we know that the integral will be invariant under pullback, that is,

$$\int_{D_2} \phi^*\omega = \int_D \omega.$$

We then calculate the pullback two-form $\phi^*\omega$:

$$\begin{aligned}\phi^*\omega =& e^{r^4}(\det J_\phi) dr \wedge d\theta \\ =& re^{r^4}\ dr \wedge d\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_D \omega =& \int_{D_2} re^{r^4}\ dr \wedge d\theta \\ =& \int_0^1 \int_0^{2\pi} re^{r^4}\ d\theta dr \\ =& \int_0^1 re^{r^4}\ [\theta]_{\theta=0}^{\theta=2\pi}\ dr \\ =& 2\pi \int_0^1 re^{r^4}\ dr.\end{aligned}$$

5. Consider the two-form $\omega = xy\, dx \wedge dy$, and let D be the region bounded by the four curves $y = \frac{1}{x}$, $y = \frac{4}{x}$, $y = x$ and $y = 4x$, with canonical orientation. Evaluate the integral of ω over D using the change of variables

$$(x, y) = \left(\frac{u}{v}, v \right).$$

Solution. The region D is shown in the figure below:

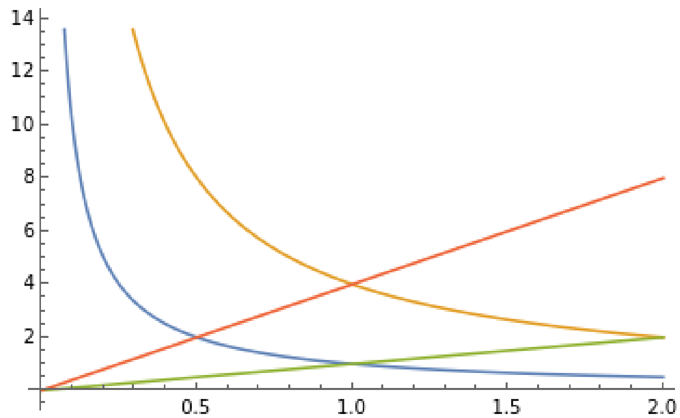


Figure 5.3.10 The region D is the region bounded by the three curves shown above. The blue curve is the curve $y = 1/x$, the orange curve is $y = 4/x$, the green line is $y = x$, and the red line is $y = 4x$.

We could evaluate the integral of ω over D by splitting the region D into two sub-regions, and then realizing these sub-regions as x-supported or y-supported. Or, we can do a change of variables, as specified in the question.

We consider the change of variables $\phi : D_2 \to D$ with

$$\phi(u, v) = \left(\frac{u}{v}, v \right).$$

What is the domain D_2 in the (u, v)-plane such that $\phi(D_2) = D$? In the variables u and v, the four curves $y = 1/x$, $y = 4/x$, $y = x$ and $y = 4x$ become

$$v = \frac{v}{u}, \qquad v = \frac{4v}{u}, \qquad , v = \frac{u}{v}, \qquad v = 4\frac{u}{v}.$$

Those four bounding equations can be rewritten as

$$u = 1, \qquad u = 4, \qquad v = \sqrt{u}, \qquad v = 2\sqrt{u}.$$

Here we used the fact that v is positive, since $y = v$ is positive. So we can describe the region D_2 in the (u, v)-plane as the v-supported region

$$D_2 = \{(u, v) \in \mathbb{R}^2 \mid u \in [1, 4], \sqrt{u} \leq v \leq 2\sqrt{u}\}.$$

This region is shown in the figure below:

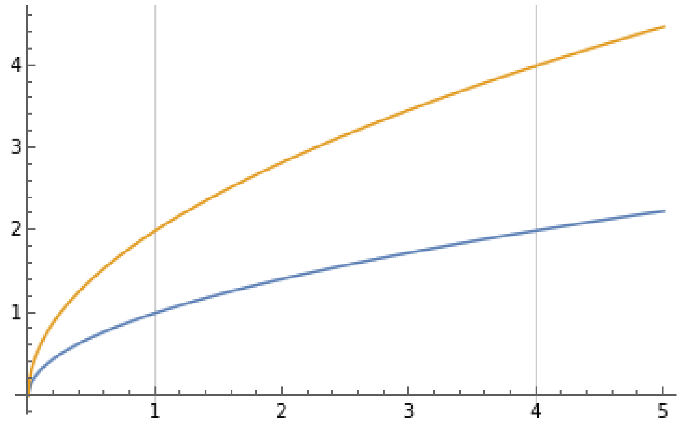


Figure 5.3.11 The region D_2 in the (u, v)-plane is the region bounded by the four curves shown above. The blue curve is the curve $v = \sqrt{u}$, the orange curve is $v = 2\sqrt{u}$, and the two grey vertical lines are $u = 1$ and $u = 4$.

Next, we calculate the determinant of the Jacobian of ϕ. We get:

$$\begin{aligned}\det J_\phi =& \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \\ =& \det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix} \\ =& \frac{1}{v}.\end{aligned}$$

As $v > 0$ on D_2, the determinant of the Jacobian is positive on D_2. So we know that the integral is invariant under pullback:

$$\int_D \omega = \int_{D_2} \phi^* \omega.$$

The pullback is:

$$\begin{aligned}\phi^*\omega =& u(\det J_\phi)\ du \wedge dv \\ =& \frac{u}{v}\ du \wedge dv.\end{aligned}$$

We then integrate:

$$\begin{aligned}\int_{D_2} \phi^*\omega &= \int_{D_2} \frac{u}{v}\ du \wedge dv \\ &= \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} \frac{u}{v}\ dvdu \\ &= \int_1^4 u \left[\ln(v)\right]_{v=\sqrt{u}}^{v=2\sqrt{u}}\ du \\ &= \int_1^4 u \left(\ln(2\sqrt{u}) - \ln(\sqrt{u})\right)\ du\end{aligned}$$

$$\begin{aligned}
&= \ln(2) \int_1^4 u \ du \\
&= \ln(2) \left(\frac{16}{2} - \frac{1}{2} \right) \\
&= \frac{15 \ln(2)}{2}.
\end{aligned}$$

For fun, let us show that we would get the same thing by evaluating the integral of $\omega = xy\ dx \wedge dy$ directly by splitting the region into two recursively supported sub-regions. Looking at the graph in Figure 5.3.10, one choice is to split D into the two x-supported regions D_1 and D_2, with

$$\begin{aligned}
D_1 =& \{(x, y) \in \mathbb{R}^2 \mid x \in [1/2, 1], 1/x \leq y \leq 4x\}, \\
D_2 =& \{(x, y) \in \mathbb{R}^2 \mid x \in [1, 2], x \leq y \leq 4/x\}.
\end{aligned}$$

We can then evaluate both integrals separately and add them up to get the result. For D_1, we calculate:

$$\begin{aligned}
\int_{D_1} \omega &= \int_{1/2}^1 \int_{1/x}^{4x} xy\ dydx \\
&= \int_{1/2}^1 x \left[\frac{y^2}{2} \right]_{y=1/x}^{y=4x} dx \\
&= \int_{1/2}^1 x \left(8x^2 - \frac{1}{2x^2} \right) dx \\
&= \left[2x^4 - \frac{1}{2} \ln(x) \right]_{x=1/2}^{x=1} \\
&= \frac{15}{8} - \frac{1}{2} \ln(2).
\end{aligned}$$

As for D_2, we get:

$$\begin{aligned}
\int_{D_2} \omega &= \int_1^2 \int_x^{4/x} xy\ dydx \\
&= \int_1^1 x \left[\frac{y^2}{2} \right]_{y=x}^{y=4/x} dx \\
&= \int_1^1 x \left(\frac{8}{x^2} - \frac{x^2}{2} \right) dx \\
&= \left[8 \ln(x) - \frac{x^4}{8} \ln(x) \right]_{x=1}^{x=2} \\
&= 8 \ln(2) - \frac{15}{8}.
\end{aligned}$$

Adding those two integrals, we get:

$$\int_D \omega = \int_{D_1} \omega + \int_{D_2} \omega$$

$$=\frac{15}{8}-\frac{1}{2}\ln(2)+8\ln(2)-\frac{15}{8}$$
$$=\frac{15\ln(2)}{2}.$$

This is the same answer that we obtained previously via our change of variables. Great!

5.4 Parametric surfaces in $\mathbb{R}^n$

Step 3: we define parametric surfaces in $\mathbb{R}^n$. This is needed so that we can define integration of two-forms over surfaces in $\mathbb{R}^n$ via pullback. The definition of parametric surfaces is similar to parametric curves, but there are new subtelties that appear, in particular in relation to the induced orientation on the surface.

Objectives

You should be able to:

- Define parametric surfaces in $\mathbb{R}^n$.
- Define the tangent plane to a parametric surface in $\mathbb{R}^n$ at a point.
- Use different parametrizations for the same surface.

5.4.1 Parametric surfaces in $\mathbb{R}^n$

Definition 5.4.1 Parametric surfaces in $\mathbb{R}^n$. Let $D \subset \mathbb{R}^2$ be a closed, bounded, and simply connected region. Let ∂D be its boundary, which is a simple closed curve. A **parametric surface** in $\mathbb{R}^n$ is a vector-valued function:

$$\begin{aligned}\alpha : D &\to \mathbb{R}^n\\ (u,v) &\mapsto \alpha(u,v) = (x_1(u,v),\ldots,x_n(u,v))\end{aligned}$$

such that:

1. α can be extended to a C^1-function on an open subset $U \subseteq \mathbb{R}^2$ that contains D;
2. The tangent vectors
$$\mathbf{T}_u = \frac{\partial \alpha}{\partial u} = \left(\frac{\partial x_1}{\partial u},\ldots,\frac{\partial x_n}{\partial u}\right), \qquad \mathbf{T}_v = \frac{\partial \alpha}{\partial v} = \left(\frac{\partial x_1}{\partial v},\ldots,\frac{\partial x_n}{\partial v}\right)$$
are linearly independent on the interior of D. (If $n=3$, this means that $\mathbf{T}_u \times \mathbf{T}_v \neq 0$ on the interior of D.)
3. If $\alpha(u_1,v_1) = \alpha(u_2,v_2)$ for any two distinct $(u_1,v_1),(u_2,v_2) \in D$, then $(u_1,v_1),(u_2,v_2) \in \partial D$. In other words, α is injective everywhere except possibly on the boundary of D.

The image $S = \alpha(D)$ is a two-dimensional subspace of $\mathbb{R}^n$, which is the surface itself. We say that the parametric surface is **smooth** if α can be extended to a smooth function on an open subset $U \subseteq \mathbb{R}^2$ containing D. ◊

This definition is obviously very similar to the definition of parametric curves in Definition 3.2.1. The three properties play similar roles: together, they ensure that the image surface S really looks like we expect a surface to look like, that is, some sort of sheet of rubber that may have been bent, deformed, warped, stretched, what not, but without introducing any puncture or tear. Property 3 ensures that our parametrization covers the image surface S exactly once (except possibly for boundary points). Property 2 also ensures that a parametrization induces a well defined choice of orientation on the image curve S, although this is more subtle than for parametric curves, as we will see in Section 5.5.

As for parametric curves, we can distinguish between two types of parametric surfaces, depending on whether the image surface is closed or not. It is a bit more subtle than for parametric curves, so let us look again at what it means for a parametric curve to be closed. In Definition 3.2.2, we said that a parametric curve $\alpha : [a, b] \to \mathbb{R}^n$ is closed if the image curve $C = \alpha([a, b])$ has no endpoints (it is a loop). If it is not closed, then we define the boundary $\partial C = \{\alpha(a), \alpha(b)\}$ as containing the endpoints of the image curve.

One way to think about a closed curve is that it is the boundary of a surface. In fact, one could say that a curve in $\mathbb{R}^n$ is closed if and only if it forms the boundary of a surface. If it is not closed, then it must have endpoints, and those form the boundary of the image curve. This definition generalizes naturally to surfaces.

Definition 5.4.2 Closed parametric surfaces. Let $\alpha : D \to \mathbb{R}^n$ be a parametric surface with image surface $S = \alpha(D) \subset \mathbb{R}^n$. We say that the parametric surface is **closed** if and only if the image surface S is the boundary of a solid in $\mathbb{R}^n$. If it is not closed, then it must have edges: we call the set ∂S consisting of all the points on the edges of S the **boundary** of the surface. $\Diamond$

Remark 5.4.3 Given a parametric surface $\alpha : D \to \mathbb{R}^n$, one should not confuse ∂D, the boundary of the closed bounded region (the domain of α), with ∂S, the boundary of the image surface. On the one hand, ∂D is never empty, as the domain of α is a closed bounded domain -- in fact, ∂D is a simple closed curve in $\mathbb{R}^2$ as we assume that D is simply connected. On the other hand, ∂S may or may not be empty, depending on whether the image surface is closed or not. Thus, generally, $\alpha(\partial D) \neq \partial S$. However, from the definition of parametric curves it follows that $\partial S \subseteq \alpha(\partial D)$; the points on the edges of S can only come from images of points on the boundary of the domain D.

Example 5.4.4 The graph of a function in $\mathbb{R}^3$. A large number of surfaces in $\mathbb{R}^3$ can be obtained as the graph of a function $f(x, y)$ of two variables:

$$z = f(x, y).$$

This defines a surface in $\mathbb{R}^3$. If we choose $(x, y) \in D$ for some closed, bounded, simply connected domain D, then we can realize the surface as a parametric surface $\alpha : D \to \mathbb{R}^3$ with

$$\alpha(u, v) = (u, v, f(u, v)).$$

Assuming that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function, we can check that this satisfies the properties of a parametric surface. Property one is automatically satisfied, as f is C^1. The tangent vectors are

$$\mathbf{T}_u = \left(1, 0, \frac{\partial f}{\partial u}\right), \qquad \mathbf{T}_v = \left(0, 1, \frac{\partial f}{\partial v}\right).$$

Those are clearly linearly independent, regardless of f. Finally, α is certainly injective on the interior of D; in fact, it is injective everywhere on D.

Since α is injective everywhere on D, including its boundary, this means that the image surface is not closed. In this case, its boundary $\partial S = \alpha(\partial D)$ is the image of the boundary of the domain. Basically, the map α takes the domain D and simply deform it continuously in the z-direction. □

Surfaces realized as the graph of a function, as in the previous example, are very easy to study parametrically. But many surfaces do not arise in this way, and may instead be given by an implicit equation in $\mathbb{R}^n$. Finding a parametrization then becomes more difficult.

Example 5.4.5 The sphere. As a second example, we consider the sphere of fixed radius R centered at the origin in $\mathbb{R}^3$. Its equation is

$$x^2 + y^2 + z^2 = R^2.$$

We cannot think of this as the graph of a function as in the previous example, because we cannot solve for z. So we need to think a bit more to realize it as a parametric surface.

As this is a sphere, it is natural that spherical coordinates may be useful. A point on the sphere radius one can be written as

$$(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) = (R\sin(\theta)\cos(\phi), R\sin(\theta)\sin(\phi), R\cos(\theta)).$$

It is easy to see that

$$x(\theta,\phi)^2 + y(\theta,\phi)^2 + z(\theta,\phi)^2 = R^2,$$

and thus those points lie on the sphere. Geometrically, θ is the inclination angle from the z direction, and ϕ is the azimuth angle measured counterclockwise from the x-axis. The inclination angle runs from 0 to π, while the azimuth angle runs from 0 to 2π.

Therefore, a realization of the sphere as a parametric surface is $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(\theta,\phi) \in \mathbb{R}^2 \mid \theta \in [0,\pi], \phi \in [0,2\pi]\},$$

and

$$\alpha(\theta,\phi) = (R\sin(\theta)\cos(\phi), R\sin(\theta)\sin(\phi), R\cos(\theta)).$$

We can check that it satisfies the properties of parametric surfaces. First, α is smooth on $\mathbb{R}^2$, so Property 1 is fine. Second, it is easy to see geometrically that α is injective everywhere, except at these points:

- For any two $\phi_1, \phi_2 \in [0,2\pi]$, $\alpha(0,\phi_1) = \alpha(0,\phi_2) = (0,0,R)$. All those points are mapped to the north pole of the sphere.

- For any two $\phi_1, \phi_2 \in [0,2\pi]$, $\alpha(\pi,\phi_1) = \alpha(\pi,\phi_2) = (0,0,-R)$. All those points are mapped to the south pole of the sphere.

- For any $\theta \in [0,\pi]$, $\alpha(\theta,0) = \alpha(\theta,2\pi) = (R\sin(\theta), 0, R\cos(\theta))$.

The important point is that these points where α is not injective are all on the boundary of the rectangular region D. Therefore Property 3 is satisfied.

As for Property 2, the tangent vectors are

$$\mathbf{T}_\theta =(R\cos(\theta)\cos(\phi), R\cos(\theta)\sin(\phi), -R\sin(\theta)),$$
$$\mathbf{T}_\phi =(-R\sin(\theta)\sin(\phi), R\sin(\theta)\cos(\phi), 0).$$

Are those linearly independent? Since the z-coordinate of $\mathbf{T}_\phi$ is zero, the only place where it could be a multiple of $\mathbf{T}_\theta$ is when the z-coordinate of $\mathbf{T}_\theta$ is also zero, that is $-R\sin(\theta) = 0$. This will only occur at $\theta = 0, \pi$, which are on the boundary of D. So we conclude that the tangent vectors must be linearly independent on the interior of D. Alternatively, we could have calculated the cross-product $\mathbf{T}_\theta \times \mathbf{T}_\phi$, and showed that it does not vanish on the interior of D.

Finally, we note that the image surface, which is the sphere, is closed, since it is the boundary of a three-dimensional solid (the ball consisting of the interior of the sphere and its boundary). □

Example 5.4.6 The cylinder. Consider the lateral surface of a cylinder of fixed radius R extending in the z-direction. Suppose that we look at the part of the cylinder from $z = 0$ to $z = 2$ (we only look at the lateral surface of the cylinder, we do not include the top and the bottom). How do we realize it as a parametric surface?

The equation of the cylinder is

$$x^2 + y^2 = R^2,$$

with z running from 0 to 2. To parametrize it, we introduce polar coordinates. Then a point on the cylinder can be written as

$$S(x(\theta, w), y(\theta, w), z(\theta, w)) = (R\cos(\theta), R\sin(\theta), w).$$

If we restrict w from 0 to 2, we get the parametric surface $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(\theta, w) \in \mathbb{R}^2 \mid \theta \in [0, 2\pi], w \in [0, 2]\},$$

and

$$\alpha(\theta, w) = (R\cos(\theta), R\sin(\theta), w).$$

Does it satisfy the properties of a parametric surface? First, α is smooth on $\mathbb{R}^2$, so Property 1 is satisfied. As for Property 2, α is injective except at the points $\alpha(0, w) = \alpha(2\pi, w) = (R, 0, w)$, for all $w \in [0, 2]$. But those are on the boundary of the rectangular region D, so it is fine. Finally, the tangent vectors are

$$\mathbf{T}_\theta = (-R\sin(\theta), R\cos(\theta), 0), \qquad \mathbf{T}_w = (0, 0, 1).$$

Those are clearly linearly independent everywhere, so Property 3 is satisfied.

The image surface here (the cylinder) is not closed; its boundary consists of its two edges, namely the circle at $w = 0$ and the circle at $w = 2$. □

5.4.2 Grid curves

In order to help visualize a parametric surface $\alpha : D \to \mathbb{R}^n$, it is sometimes useful to sketch the grid curves on the image surface $S = \alpha(D)$. Suppose that D is a closed, simply connected, bounded domain in the uv-plane. The idea is to consider the horizontal and vertical lines in the uv-plane that lie on D; those are the lines $u = u_0$ or $v = v_0$ for constant u_0, v_0. The image of these lines under the map α will be curves on the image surface S: those are called the **grid curves**. They help visualize how the parametrization α maps the region D onto the image surface S.

Example 5.4.7 Grid curves on the sphere. Consider the sphere of Example 5.4.5, which is realized as the parametric surface $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(\theta, \phi) \in \mathbb{R}^2 \mid \theta \in [0, \pi], \phi \in [0, 2\pi]\},$$

and

$$\alpha(\theta, \phi) = (R\sin(\theta)\cos(\phi), R\sin(\theta)\sin(\phi), R\cos(\theta)).$$

The domain D is a rectangular region. Horizontal lines on D correspond to lines with $\phi = C$ for constants C. The image of the horizontal lines would be the grid curves

$$\alpha(\theta, C) = (R\cos(C)\sin(\theta), R\sin(C)\sin(\theta), R\cos(\theta)).$$

Since θ is the inclination angle, those curves correspond to curves of constant longitude, starting at the north pole and ending at the south pole.

As for the vertical lines on D, they are given by the lines with $\theta = K$ for constants K. The image of the vertical lines would be the grid curves

$$\alpha(K, \phi) = (R\sin(K)\cos(\phi), R\sin(K)\sin(\phi), R\cos(K)).$$

Those correspond to the curves of constant latitude, going all around the sphere. □

5.4.3 The tangent planes

Let $\alpha : D \to \mathbb{R}^n$ be a parametric surface, and $S = \alpha(D) \subset \mathbb{R}^n$ the image curve. In the definition of parametric surfaces Definition 5.4.1, we introduced the "tangent vectors"

$$\mathbf{T}_u = \frac{\partial \alpha}{\partial u} = \left(\frac{\partial x_1}{\partial u}, \ldots, \frac{\partial x_n}{\partial u}\right), \qquad \mathbf{T}_v = \frac{\partial \alpha}{\partial v} = \left(\frac{\partial x_1}{\partial v}, \ldots, \frac{\partial x_n}{\partial v}\right),$$

but we did not really explain their geometric meaning.

At a point $P \in S$, there is a two-dimensional space of tangent directions to the surface. This is what is called the "tangent plane" to the surface at $P \in S$. It gives the best linear approximation of the surface at that point.

The claim is that the tangent plane at a point $p \in S$ is the two-dimensional vector space spanned by the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$. Indeed, recall that the grid curves are the curves $\alpha(u, C) \in S$ and $\alpha(K, v) \in S$ for constant C, K. By definition of partial derivatives, it then follows that the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent vectors at the point p pointing in the direction of these grid curves. Furthermore, in the definition of parametric surfaces, we assume that $\mathbf{T}_u$ and $\mathbf{T}_v$ are linearly independent on the interior of D. Thus, for any point $p \in S$ which is in the image of the interior of D, we know that the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ span a two-dimensional vector space. We can thus define the tangent planes for a parametric surface as follows:

Definition 5.4.8 Tangent planes to a parametric surface. Let $\alpha : D \to \mathbb{R}^n$ be a parametric surface, with image surface $S = \alpha(D)$. For any point $p \in S$ which is in the image of the interior of D, we define the **tangent plane** T_pS **at** p to be the two-dimensional vector space spanned by the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ at p. It provides the best linear approximation of the surface at that point. $\Diamond$

If we now restrict to parametric surfaces in $\mathbb{R}^3$, then there is another way that we can specify the tangent plane at a point. We can always find the equation of a plane in $\mathbb{R}^3$ by specifying a point on the plane and a normal (perpendicular) vector to the plane. Indeed, if $\mathbf{n}$ is a vector normal to a plane, and (x_0, y_0, z_0) is a point on the plane, then the equation of the plane is:

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Thus, given a parametric surface $\alpha : D \to \mathbb{R}^3$ with image surface $S = \alpha(D)$, at a point $p \in S$ we can specify the tangent plane by instead specifying the normal vector to the surface at that point. But what is the normal vector? Well, if $\mathbf{T}_u$ and $\mathbf{T}_v$ are both tangent vectors, then we know how to find a new vector that is perpendicular to both vectors: we take the cross-product! We get:

Definition 5.4.9 Normal vectors to a parametric surface in $\mathbb{R}^3$. Let $\alpha : D \to \mathbb{R}^n$ be a parametric surface, with image surface $S = \alpha(D)$. For any point $p \in S$ which is in the image of the interior of D, we define the **normal vector n at** p to be the vector:

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v.$$

We note here that this normal vector is not normalized, i.e. it does not have length one. To get a normalized vector we would divide by its norm. When we talk about the normalized normal vector later on, we will use the notation

$$\hat{\mathbf{n}} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|},$$

to avoid ambiguity. $\Diamond$

5.4.4 Exercises

1. Realize the part of the plane $z = x - 2$ that lies inside the cylinder $x^2 + y^2 = 4$ as a parametric surface.

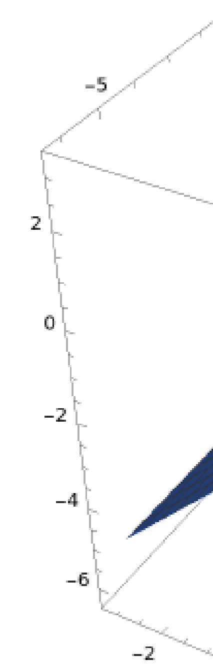


Figure 5.4.10 The surface S is
$x^2 + y^2 = 4$ (in orange).

Solution. Let us first sketch a picture of what the surface should look like:

The surface S is the part of the blue plane that lies within the orange cylinder. How can we realize it as a parametric surface?

We first, we know how to parametrize the plane $z = x - 2$. The function

$$\alpha(u, v) = (u, v, u - 2)$$

is a parametrization of the plane. What we need to determine now is what is the region D in the (u, v)-plane such that $\alpha(D) = S$, where S is the part of the plane contained within the cylinder $x^2 + y^2 = 4$. What we can do is find the boundary curve of the surface S, which consists in the intersection of the plane $z = x - 2$ and the cylinder $x^2 + y^2 = 4$. Using our parametrization above $\alpha(u, v)$, we see that $x^2 + y^2 = 4$ corresponds to the equation $u^2 + v^2 = 4$. In other words, if we define D to be the disk of radius 2 centered at the origin, then α maps its boundary (the circle of radius two) to the boundary of the surface S, and the interior to the interior. So this gives an appropriate choice of domain D. We can describe D as a u-supported region:

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [-2, 2], -\sqrt{4 - u^2} \leq v \leq \sqrt{4 - u^2}\}.$$

Then the surface S is realized as the parametric surface $\alpha : D \to \mathbb{R}^3$, with D above and $\alpha(u, v) = (u, v, u - 2)$.

Note that there are many other parametrizations that we could have used. For instance, since the region D is a disk of radius 2, we could have used polar coordinates. In other words, if we do the change of coordinates $u = r\cos(\theta)$, $v = r\sin(\theta)$, the region becomes

$$D_2 = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,2], \theta \in [0,2\pi],$$

and $\alpha_2 : D_2 \to \mathbb{R}^3$ with

$$\alpha_2(r,\theta) = (r\cos(\theta), r\sin(\theta), r\cos(\theta) - 2).$$

That is another parametrization of the same surface S.

2. Realize the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2+y^2}$ as a parametric surface.

Solution. Let us first sketch a picture of what the surface should look like:

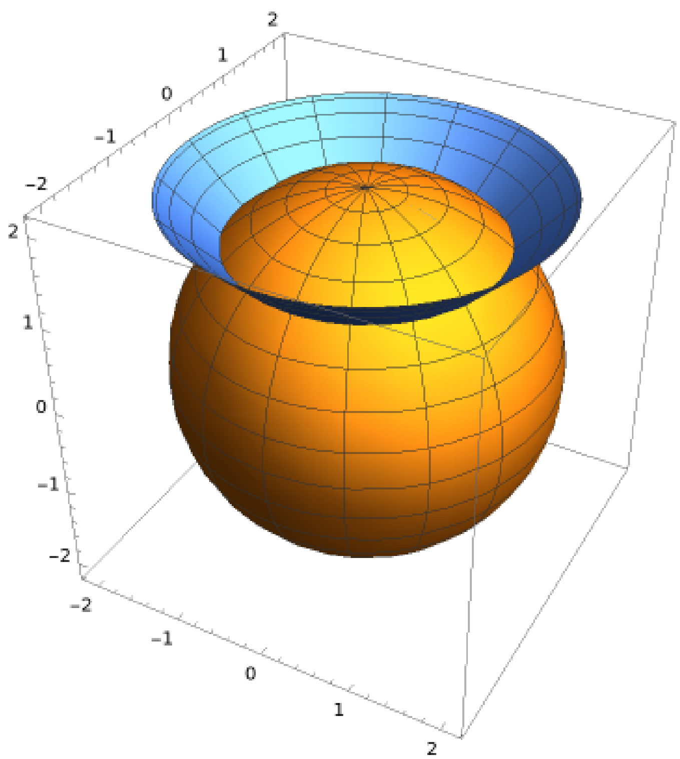


Figure 5.4.11 The surface S is the part of the sphere with radius 2 (in orange) that lies above the cone in blue.

How can we realize S as a parametric surface?

We start by parametrizing the sphere of radius 2 centered at the origin. We use spherical coordinates. A parametrization for the sphere is $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(\theta,\phi) \in \mathbb{R}^2 \mid \theta \in [0,\pi], \phi \in [0,2\pi]\},$$

and

$$\alpha(\theta,\phi) = (2\sin(\theta)\cos(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\theta)).$$

This is a parametrization of the sphere, but this is not the surface we are interested in. We only want to keep the part of the sphere that lies above the cone $z = \sqrt{x^2+y^2}$. In other words, we want to restrict the range of the inclination angle θ so that it only goes from 0 to the angle where the cone intersects the sphere. What is this angle? The cone has equation $z = \sqrt{x^2+y^2}$. Using our parametrization above for points on the sphere, we see that the points on the sphere that also lie on the cone (i.e. at the intersection of both surfaces) must satisfy

$$\begin{aligned} 2\cos(\theta) =& \sqrt{(2\sin(\theta)\cos(\phi))^2 + (2\sin(\theta)\sin(\phi))^2} \\ =& 2\sin(\theta), \end{aligned}$$

where we used the fact that $\sin(\theta) \geq 0$ since $\theta \in [0,\pi]$. Therefore, we must have

$$\tan(\theta) = 1.$$

The only solution with $\theta \in [0,\pi]$ is $\theta = \pi/4$.

Therefore, we conclude that the region of the sphere that lies above the cone will be given by restricting the inclination angle to be between 0 and $\pi/4$. More precisely, a parametrization of our surface is given by $\alpha_2 : D_2 \to \mathbb{R}^3$, with

$$D_2 = \{(\theta,\phi) \in \mathbb{R}^2 \mid \theta \in [0,\pi/4], \phi \in [0,2\pi]\},$$

and

$$\alpha_2(\theta,\phi) = (2\sin(\theta)\cos(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\theta)).$$

3. Consider the parametric surface $\alpha : D \to \mathbb{R}^3$ with $D = \{(u,v) \in \mathbb{R}^2 \mid u \in [0,2], v \in [0,3]\}$ and

$$\alpha(u,v) = (u, v^3+1, u+v).$$

(a) Find the tangent vectors $\mathbf{T}_u$, $\mathbf{T}_v$, and the normal vector $\mathbf{n}$.

(b) Find an equation for the tangent plane to the image surface $\alpha(D)$ at the point $(1,2,2)$.

Solution. (a) We calculate the tangent vectors by taking partial derivatives:

$$\mathbf{T}_u = \frac{\partial \alpha}{\partial u} = (1,0,1), \qquad \mathbf{T}_v = \frac{\partial \alpha}{\partial v} = (0,3v^2,1).$$

We find the normal vector by taking the cross-product of the tangent vectors:

$$\begin{aligned} \mathbf{n} =& \mathbf{T}_u \times \mathbf{T}_v \\ =& \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 3v^2 & 1 \end{pmatrix} \\ =& (-3v^2, -1, 3v^2). \end{aligned}$$

(b) To find an equation of the tangent plane at the point $(1,2,2)$, we use the point-normal form for the equation of a plane. First, we see that $\alpha(1,1) = (1,2,2)$, so the point is on the image surface $\alpha(D)$ for the values of the parameters $(u, v) = (1, 1)$. The normal vector to the plane at that point is $\mathbf{n}(1, 1) = (-3, -1, 3)$. Then, by the point-normal form, we know that the equation of the tangent plane is

$$\mathbf{n} \cdot (x - 1, y - 2, z - 2) = (-3, -1, 3) \cdot (x - 1, y - 2, z - 2) = 0.$$

Evaluating the dot product, we get the equation of the tangent plane:

$$-3x - y + 3z = 1.$$

4. Consider the curve $y = x^2$ with $x \in [0, 2]$. Find a parametrization for the surface obtained by rotating the curve about the y-axis.

Solution. First, we sketch the curve $y = x^2$ in the (x, y)-plane inside $\mathbb{R}^3$:

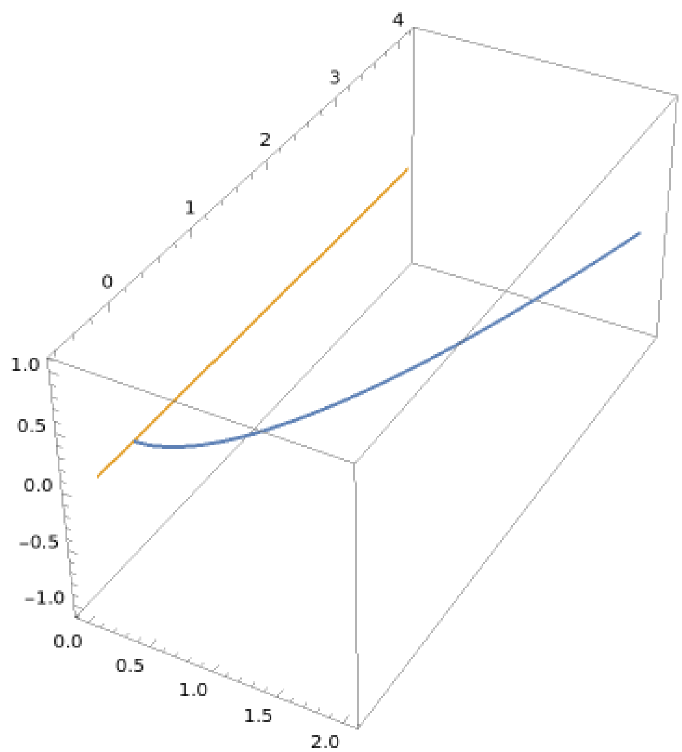


Figure 5.4.12 The curve $y = x^2$ in the (x, y)-plane inside $\mathbb{R}^3$ is shown in blue; the orange line is the axis of rotation, which is the y-axis.

We rotate the curve about the y-axis, which is the orange line. After rotation, we get the following surface:

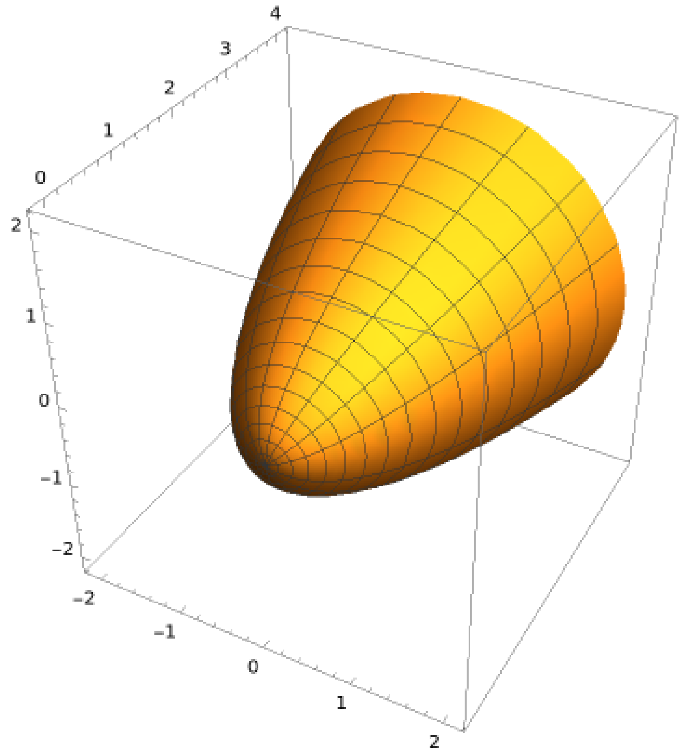


Figure 5.4.13 The surface obtained by rotating the curve $y = x^2$ about the y-axis.

How do we parametrize this surface? Let's think about it. First, we can parametrize the curve $y = x^2$ in the (x, y)-plane within $\mathbb{R}^3$, with $x \in [0, 2]$, by $\phi(t) = (t, t^2, 0)$ with $t \in [0, 2]$. What happens if we rotate the curve about the y-axis? For a fixed value of t, the point with (x, z)-coordinates $(t, 0)$ gets rotated about the y-axis on a circle with radius t. We can thus parametrize this circle by $(x, z) = (t\cos(\theta), t\sin(\theta))$, with $\theta \in [0, 2\pi]$. We do that for all values of $t \in [0, 2]$, and we end up with the surface of revolution. The resulting parametrization is $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(t, \theta) \in \mathbb{R}^2 \mid t \in [0, 2], \theta \in [0, 2\pi]\}$$

and

$$\alpha(t, \theta) = (t\cos(\theta), t^2, t\sin(\theta)).$$

5.5 Orientation of parametric surfaces in $\mathbb{R}^3$

Step 3, part II: we show that the parametrization of a surface induces an orientation on the image surface. However, this is a bit more subtle than for parametric curves, as not all surfaces are orientable. Thus we first discuss orientability of surfaces, and then show that the parametrization of an orientable surface induces an orientation on the image surface. We now focus on surfaces in $\mathbb{R}^3$, which are easier to visualize.

Objectives

You should be able to:

- Define orientable and non-orientable surfaces.
- Determine the orientation of a parametric surface in $\mathbb{R}^3$.
- Relate the orientation of a parametric surface in $\mathbb{R}^3$ to the normal vector.

5.5.1 Orientable and non-orientable surfaces

In Section 5.2 we defined the orientation of $\mathbb{R}^n$ and of a closed bounded region therein. We now want to define the orientation of a surface in $\mathbb{R}^n$. We will focus on a surface in $\mathbb{R}^3$ in this section, since it is easier to visualize.

Recall from Definition 5.2.1 that the orientation of the vector space $\mathbb{R}^n$ is given by a choice of "twirl", specified by a choice of ordered basis on $\mathbb{R}^n$. For $\mathbb{R}^2$, this amounts to specifying a direction of rotation: either counterclockwise or clockwise. The canonical orientation is counterclockwise. The orientation of a closed, bounded, and simply connected region $D \subset \mathbb{R}^n$ is the orientation induced by a choice of ordered basis on the ambient space $\mathbb{R}^n$.

Now suppose that $\alpha : D \to \mathbb{R}^3$ is a parametric surface, with image surface $S = \alpha(D) \subset \mathbb{R}^3$. We would like to define an orientation on S in a way that is similar to what we did for the plane (or for $D \subset \mathbb{R}^2$) — we want to define a "direction of rotation" on the surface S. But this is not so obvious anymore, since S is a like a rubber sheet that can be bent, stretched, deformed, wrapped, what not. So what do we mean by direction of rotation?

The concept of tangent plane introduced in Definition 5.4.8 comes to the rescue. At each point $p \in S$ (not on its boundary), the tangent plane T_pS is a two-dimensional vector space, i.e. $\mathbb{R}^2$. So we can naturally define a direction of rotation on the tangent plane at $p \in S$ by specifying an ordered basis $\{\mathbf{T}_u, \mathbf{T}_v\}$. Then, we can define the orientation of the surface S by assigning an ordered basis to all tangent planes in a continuous manner.

When the surface is in $\mathbb{R}^3$, there is an equivalent way of defining a direction of rotation on the tangent planes. Instead of specifying an ordered basis for the tangent planes T_pS, we can specify a normal vector, as in Definition 5.4.9. If $\mathbf{T}_u$ and $\mathbf{T}_v$ form an ordered basis for the tanget plane T_pS at a point $p \in S$, then we define the normal vector

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v.$$

The ordering is important here, since this is what gives the direction of rotation. Changing from a direction of rotation to the opposite one amounts to exchanging the order of the two basis vectors, which, in turn, sends $\mathbf{n}$ to $-\mathbf{n}$. So what matters here is the direction of the normal vector: it either points in one direction or the opposite direction, and this defines the two choices of orientation on the tangent plane at this point.

So we can think of an orientation of a surface as an assignment of a normal vector at all points on S in a continuous manner. Since what matters is the direction of the normal vector, choosing an assignment of a normal vector at all points on S is basically the same as choosing a side for the surface S. Which raises the following question: **are all surfaces in $\mathbb{R}^3$ orientable?** That is, do all surfaces in $\mathbb{R}^3$ have two sides?

Perhaps surprisingly, the answer is no! Not all surfaces are orientable. There exists surfaces that only have one side! We will come back to this in a second. Let us now define more carefully the notion of an "orientable surface".

Definition 5.5.1 Orientable surfaces and orientation. Let $S \subset \mathbb{R}^3$ be a surface. We say that S is **orientable** if there exists a continuous function $\mathbf{n} : S \to \mathbb{R}^3$ which assigns to all points $p \in S$ (not on the boundary) the normal vector to the tangent plane T_pS. We say that it is **non-orientable** otherwise.

If S is orientable, then an **orientation** is a choice of continuous function $\mathbf{n} : S \to \mathbb{R}^3$. It assigns to all points on S a normal vector, which basically specifies one side of the surface (and also determines a direction of rotation on the tangent planes). Saying that the assignment of the normal vector is continuous amounts to saying that you pick the same side all around the surface — you do not suddenly jump from one side to the other. $\Diamond$

Most surfaces that we encounter in real life, such as spheres, cylinders, planes, tori, etc. are orientable. But the prototypical example of a non-orientable surface is the **Möbius strip**. This is a strange surface, shown in Figure 5.5.2. (It can be constructed by taking a long rectangle of paper, giving it one half-twist, and then taping back the ends together.) If you start at a point on the surface, you can pick a side, which amounts to defining a normal vector. Then you can move around the surface, always picking the same side. But at some point you will be back at the point you started with, but on the other side! (Try it!) So the assignment of normal vectors is not continuous, since you if you stopped just before the point you started with, the normal vector would suddenly have to jump from one side to the other.

What is going on? The point is that, on the Möbius strip, by doing this assignment of normal vectors, you end up labeling both sides of the strip. The reason is: the strip really only has one side! Crazy.

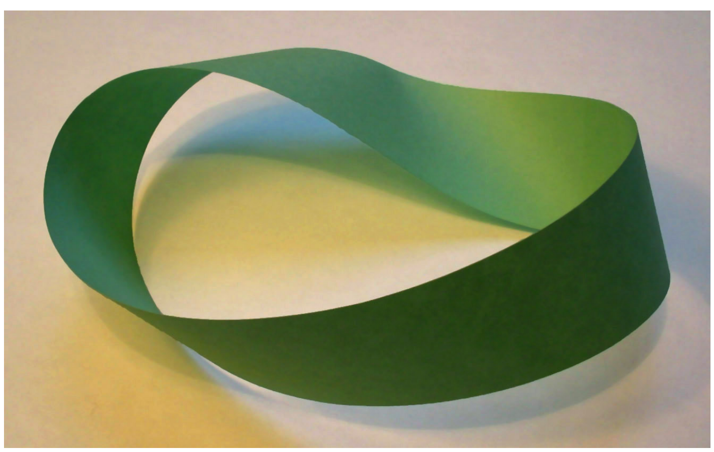


Figure 5.5.2 The Möbius strip (By David Benbennick - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=50359).

The Möbius strip is very cool, but from now on we will always assume that our surfaces are orientable.

5.5.2 Orientation of a parametric surface

Now that we know how to define the orientation of a surface in $\mathbb{R}^3$, we can go back to our parametric surfaces. Are parametric surfaces naturally oriented? The answer is yes, as long as the image surface is assumed to be orientable (as we could, for instance, realize the Möbius strip as a parametric surface).

Lemma 5.5.3 Parametric surfaces are oriented. *Let*

$$\begin{aligned} \alpha : D &\to \mathbb{R}^3 \\ (u,v) &\mapsto \alpha(u,v) = (x(u,v), y(u,v), z(u,v)) \end{aligned}$$

be a parametric surface, and assume that the image surface $S = \alpha(D) \subset \mathbb{R}^3$ is orientable. The normal vector

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v,$$

with $\mathbf{T}_u$ and $\mathbf{T}_v$ the two tangent vectors from Definition 5.4.8, naturally induces an orientation on S.

Note that the order is important here: we must take the cross-product $\mathbf{T}_u \times \mathbf{T}_v$ with the two vectors in the same order as the coordinates (u,v) on $D \subset \mathbb{R}^2$.

Proof. There isn't much to prove here. We showed in Definition 5.4.8 and Definition 5.4.9 that the normal vector was well defined (and non-zero) for all points that are not in the image of the boundary of D. So this gives a continuous assignment of a normal vector, and since we assume that S is orientable, it induces an orientation on S. ■

Remark 5.5.4 As for parametric curves, there is another way of thinking about this statement. We can think of the parametrization $\alpha : D \to \mathbb{R}^3$ as not only mapping the region D, but as also mapping its orientation. When we define parametric surfaces, we always think of the domain D as being given the canonical orientation (counterclockwise), which is induced by the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ on $\mathbb{R}^2$, with $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$. The vector $\mathbf{e}_1$ points in the u-direction, and hence is mapped to $\mathbf{T}_u$, while the vector $\mathbf{e}_2$ points in the v-direction and is mapped to $\mathbf{T}_v$. So the counterclockwise orientation on D induces the orientation given by the ordered basis $\{\mathbf{T}_u, \mathbf{T}_v\}$ on the tangent planes.

Finally, in Definition 5.2.9 we discussed the induced orientation on the boundary curve ∂D of a closed bounded region D. If D has canonical orientation, the induced orientation of the boundary is the direction of motion if you walk along the boundary curve keeping the region on your left. Similarly, a parametric surface induces an orientation on the boundary ∂S of the image surface S. Since for parametric surfaces we always start with the canonical orientation on D, we expect the induced orientation on the boundary of the image surface to be defined similarly, by walking along the boundary curve keeping the surface on your left. However, things are more subtle here, since the surface is in $\mathbb{R}^3$, so it's not immediately clear what it means to walk along the boundary keeping the surface your left... In which direction should your head be pointing?

For a region $D \subset \mathbb{R}^2$, we of course assumed that you were standing up. One way to think about this is that you had your head in the positive z-direction (if you assume that D is in the xy-plane embedded in $\mathbb{R}^3$). We now know that we can think of an orientation as a choice of normal vector, and the canonical orientation on the xy-plane corresponds to the normal vector pointing in the positive z-direction. In other words, we implicitly assumed that your head was pointing in the direction of the normal vector (or that you were walking on the side chosen by the orientation). This now generalizes to surfaces in $\mathbb{R}^3$.

Definition 5.5.5 Induced orientation on the boundary of a parametric surface. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface, with image surface $S = \alpha(D)$ orientable and oriented by the parametrization. Let ∂S be the boundary (the edges) of the image surface S. The **induced orientation** on the boundary curve ∂S is the direction of travel if you keep the region on the left, with your head in the direction of the normal vector $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$. We denote by $\partial\alpha$ the boundary of the image surface with its orientation induced by the parametrization α. ♢

Example 5.5.6 Upper half-sphere. Let us realize the upper half-sphere of radius R as a parametric surface, and study its orientation and the induced orientation on its boundary.

The upper half-sphere has equation

$$x^2 + y^2 + z^2 = R^2,$$

keeping only the points with $z \geq 0$. There are many ways that we can parametrize it. For instance, we can use either Cartesian or spherical coordinates. Let us do it in spherical coordinates, and leave the Cartesian coordinates parametrization for Exercise 5.5.4.1.

We recall from Example 5.4.5 the parametrization of the sphere of radius R. A parametrization of the upper half-sphere is obtained in the same way, but restricting the inclination angle θ from 0 to $\pi/2$. We get the parametric surface $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(\theta, \phi) \in \mathbb{R}^2 \mid \theta \in \left[0, \frac{\pi}{2}\right], \phi \in [0, 2\pi]\},$$

and

$$\alpha(\theta, \phi) = (R\sin(\theta)\cos(\phi), R\sin(\theta)\sin(\phi), R\cos(\theta)).$$

Now suppose that D has canonical orientation, as always for parametric surfaces. What is the induced orientation on the upper half-sphere? We calculate the tangent vectors:

$$\begin{aligned}\mathbf{T}_\theta &= (R\cos(\theta)\cos(\phi), R\cos(\theta)\sin(\phi), -R\sin(\theta))\\ \mathbf{T}_\phi &= (-R\sin(\theta)\sin(\phi), R\sin(\theta)\cos(\phi), 0)\,.\end{aligned}$$

The normal vector is then:

$$\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_\phi = R^2\left(\sin^2(\theta)\cos(\phi), \sin^2(\theta)\sin(\phi), \sin(\theta)\cos(\theta)\right).$$

The main question is the direction of the normal vector: does it point inwards (towards the center of the sphere), or outwards? In other words, is the orientation selecting the inner surface of the upper half-sphere or the outer surface? To find out, we only need to look at what happens at a given point on the upper half-sphere (we need to pick a point that is not in the image of the boundary of D). Let's pick the point with parameters $(\theta, \phi) = (\frac{\pi}{4}, 0)$. This is the point $\frac{\sqrt{2}}{2}R\,(1, 0, 1)$. At this point, the normal vector is:

$$\mathbf{n}\left(\frac{\pi}{4}, 0\right) = \frac{R^2}{2}\,(1, 0, 1)\,.$$

We thus see that the normal vector points outwards in the radial direction, i.e. away from the origin. Therefore, the induced orientation on the upper half-sphere it outwards, i.e. the side of the surface selected is the outer surface.

What about the induced orientation on the boundary? The boundary of the upper half-sphere is the circle in the xy-plane with radius R (that's the edge of the upper half-sphere). In our parametrization, it corresponds to the points with parameters $(\theta, \phi) = (\frac{\pi}{2}, \phi)$ for $\phi \in [0, 2\pi]$. The induced orientation should correspond to the direction of motion if we walk along the circle, with our head in the direction of the normal vector, keeping the surface on your left. As the normal vector point outwards, we are walking on the circle with our head pointing away from the origin. If we keep the surface on our left, we end up walking along the circle counterclockwise in the xy-plane. This is the induced orientation on the boundary circle. $\square$

5.5.3 Orientation-preserving reparametrizations of a surface

Just as for parametric curves, there are many ways to parametrize a given surface $S \subset \mathbb{R}^3$. If we have a parametric surface $\alpha : D \to \mathbb{R}^3$, with

$$\alpha(u, v) = (x(u, v), y(u, v), z(u, v)),$$

and we think of u, v as functions of new variables s, t, then by doing this change of variable we may end up with a new parametrization of the same image surface. The question is whether this new parametrization induces the same orientation on the image surface or the opposite orientation.

Lemma 5.5.7 Orientation-preserving reparametrizations. *Let $\alpha : D_1 \to \mathbb{R}^3$ be a parametric surface, with the image surface $S = \alpha(C)$ orientable, and $\alpha(u, v) = (x(u, v), y(u, v), z(u, v))$. Let $D_2 \subset \mathbb{R}^2$ be another closed, bounded, and simply-connected region. Let $\phi : D_2 \to D_1$ be a bijective and invertible function that can be extended to a C^1-function on an open subset $U \subseteq \mathbb{R}^2$ that contains D_2, as in Definition 5.3.6. Then the pullback*

$$\begin{aligned} &\phi^*\alpha : D_2 \to \mathbb{R}^3 \\ &\qquad (s, t) \mapsto (\phi^*x(s, t), \phi^*y(s, t), \phi^*z(s, t)) = (x(\phi(s, t)), y(\phi(s, t)), z(\phi(s, t))) \end{aligned}$$

is another parametrization of the same image surface S.

*Furthermore, if $\det J_\phi > 0$, the induced orientation on S is the same for both parametrizations α and $\phi^*\alpha$, and we say that the reparametrization is **orientation-preserving**. If $\det J_\phi < 0$, the two parametrizations α and $\phi^*\alpha$ have opposite orientations, and the reparametrization is **orientation-reversing**.*

Proof. First, it is clear that $\alpha(D_1) = \phi^*\alpha(D_2)$, i.e. the image surfaces are the same, since we are simply composing maps. But we need to show that $\phi^*\alpha$ is a parametric surface, according to Definition 5.4.1.

Property 1 is clearly satisfied for $\phi^*\alpha$ since ϕ is assumed to be C^1. Property 3 is also satisfied, since ϕ is bijective. As for Property 2, let us denote the tangent vectors to $\phi^*\alpha$ by $\mathbf{V}_s$ and $\mathbf{V}_t$. By definition,

$$\mathbf{V}_s = \frac{\partial}{\partial s}\phi^*\alpha(s, t) = \left(\frac{\partial x(\phi(s, t))}{\partial s}, \frac{\partial y(\phi(s, t))}{\partial s}, \frac{\partial z(\phi(s, t))}{\partial s} \right),$$

$$\mathbf{V}_t = \frac{\partial}{\partial t}\phi^*\alpha(s,t) = \left(\frac{\partial x(\phi(s,t))}{\partial t}, \frac{\partial y(\phi(s,t))}{\partial t}, \frac{\partial z(\phi(s,t))}{\partial t}\right).$$

If we denote $\phi(s,t) = (u(s,t), v(s,t))$, using the chain rule, we get:

$$\begin{aligned}\mathbf{V}_s =& \frac{\partial u}{\partial s}\left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v}\right)\\ =& \frac{\partial u}{\partial s}\mathbf{T}_u + \frac{\partial v}{\partial s}\mathbf{T}_v,\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathbf{V}_t =& \frac{\partial u}{\partial t}\left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v}\right)\\ =& \frac{\partial u}{\partial t}\mathbf{T}_u + \frac{\partial v}{\partial t}\mathbf{T}_v.\end{aligned}$$

Calculating the cross-product, we get:

$$\begin{aligned}\mathbf{V}_s \times \mathbf{V}_t =& \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial v}{\partial s}\frac{\partial u}{\partial t}\right)\mathbf{T}_u \times \mathbf{T}_v\\ =& (\det J_\phi)\mathbf{T}_u \times \mathbf{T}_v.\end{aligned}$$

Therefore, if $\mathbf{T}_u$ and $\mathbf{T}_v$ are linearly independent, then $\mathbf{T}_u \times \mathbf{T}_v \neq 0$, and since ϕ is invertible, $\det J_\phi \neq 0$. Therefore $\mathbf{V}_s \times \mathbf{V}_t \neq 0$, and Property 2 is satisfied.

From this calculation we can also relate the induced orientations of the two parametrization α and $\phi^*\alpha$. If we denote by $\mathbf{n}$ the normal vector for α, and $\mathbf{m}$ the normal vector for $\phi^*\alpha$, from the calculation above we get:

$$\mathbf{m} = (\det J_\phi)\mathbf{n}.$$

Thus, if $\det J_\phi > 0$, then the normal vectors of both parametrizations have the same sign (they pick the same side for the image surface) and the induced orientations are the same, while if $\det J_\phi < 0$, they have opposite signs (they pick opposite sides) and the induced orientations are opposite. ■

5.5.4 Exercises

1. Redo the parametrization of the upper half-sphere of radius R, as in Example 5.5.6, but in Cartesian coordinates. (The region D in this case should be a disk of radius R.) Find the induced orientation on the image surface by calculating the normal vector.

Solution. The upper half-sphere of radius R is the surface in $\mathbb{R}^3$ defined by the equation

$$x^2 + y^2 + z^2 = R^2$$

with $z \geq 0$. Since $z \geq 0$, we can solve explicitly for z. We get:

$$z = \sqrt{R^2 - x^2 - y^2}.$$

This gives us another way of parametrizing the surface directly, since it is the graph of a function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$. We proceed as in Example 5.4.4. We find a parametrization $\alpha : D \to \mathbb{R}^3$ with

$$\alpha(u, v) = (u, v, \sqrt{R^2 - u^2 - v^2}).$$

However, we need to specify D. The region should be the disk that is at the bottom of the upper half-sphere, which is the disk of radius R centered at the origin. So the region D in the (u, v)-plane should be given by $u^2 + v^2 \leq R^2$. We can rewrite this as a u-supported region:

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [-R, R], -\sqrt{R^2 - u^2} \leq v \leq \sqrt{R^2 - u^2}.$$

To find the induced orientation, we need to calculate the normal vector. We first find the tangent vectors:

$$\begin{aligned} \mathbf{T}_u =& \frac{\partial \alpha}{\partial u} = \left(1, 0, -\frac{u}{\sqrt{R^2 - u^2 - v^2}}\right), \\ \mathbf{T}_v =& \frac{\partial \alpha}{\partial v} = \left(0, 1, -\frac{v}{\sqrt{R^2 - u^2 - v^2}}\right). \end{aligned}$$

The normal vector is given by the cross-product:

$$\begin{aligned} \mathbf{n} =& \mathbf{T}_u \times \mathbf{T}_v \\ =& \left(\frac{u}{\sqrt{R^2 - u^2 - v^2}}, \frac{v}{\sqrt{R^2 - u^2 - v^2}}, 1\right). \end{aligned}$$

We need to determine in which direction it points (outwards of the sphere, or inwards), to determine the induced orientation. Let's pick the point on the sphere with parameters $(u, v) = (0, 0)$. This is the point $\alpha(0, 0) = (0, 0, R)$, which is the north pole. The normal vector at this point is $\mathbf{n}(0, 0) = (0, 0, 1)$. It points upwards, that is, in the outwards direction (away from the origin, which is the center of the sphere). Therefore, this parametrization induces the orientation on the upper half-sphere given by a normal vector pointing outwards (it picks the outside of the surface).

We note that this is the same induced orientation as in Example 5.5.6. Indeed, we can relate the two parametrizations by the change of variables

$$(u, v) = (R \sin(\theta) \cos(\phi), R \sin(\theta) \sin(\phi)).$$

The determinant of the Jacobian of the transformation is

$$\det \begin{pmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \end{pmatrix} = \det \begin{pmatrix} R \cos(\theta) \cos(\phi) & -R \sin(\theta) \sin(\phi) \\ R \cos(\theta) \sin(\phi) & R \sin(\theta) \cos(\phi) \end{pmatrix} = R \sin(\theta) \cos(\theta),$$

which is positive for $\theta \in [0, \pi/2]$. Therefore, the two parametrizations should induce the same orientation, as we found.

2. Consider the surface S consisting of the part of the plane $z + x + y = 2$ that lies inside the cylinder $x^2 + y^2 = 1$. Find two parametrizations of S that have opposite orientation.

Solution. First, we can parametrize the plane by $\alpha(u, v) = (u, v, 2 - u - v)$. Now we need to find a region D in the (u, v)-plane such that $\alpha(D)$ is the part of the plane that lies inside the cylinder $x^2 + y^2 = 1$. We see that if we take the region D to be the disk $u^2 + v^2 = 1$, then the boundary circle of D is mapped to the boundary of the surface S, that is, the closed curve at the intersection of the plane and the cylinder. So this is the region D that we are looking for. We can realize it as a u-supported region. The final parametrization would be $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [-1, 1], -\sqrt{1 - u^2} \leq v \leq \sqrt{1 - u^2}\}$$

and

$$\alpha(u, v) = (u, v, 2 - u - v).$$

What is the induced orientation? The tangent vectors are:

$$\mathbf{T}_u = (1, 0, -1), \qquad \mathbf{T}_v = (0, 1, -1).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = (1, 1, 1),$$

which defines the induced orientation on the planar region S. (Note that it is a constant vector since the region S is planar (i.e. part of a plane), and so the normal direction is the same everywhere on S.)

How can we find a second parametrization that has the opposite orientation? Well, there are many possibilities. But one easy way to get the normal vector to have opposite sign is to exchange the order of the two tangent vectors, i.e. exchange the variables u and v. In other words, we could define the parametrization $\alpha_2 : D \to \mathbb{R}^3$ with the same region D but

$$\alpha_2(u, v) = (v, u, 2 - u - v).$$

(All we did is interchange u and v -- the image surface is obviously the same.) Then the tangent vectors are

$$\mathbf{T}_u = (0, 1, -1), \qquad \mathbf{T}_v = (1, 0, -1),$$

and the normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = (-1, -1, -1),$$

which points in the opposite direction, and hence this parametrization induces the opposite orientation.

3. We have seen in Example 5.4.4 how we can realize the graph of a function $f(x, y)$ as a parametric surface in $\mathbb{R}^3$.

(a) Show that the natural parametrization of Example 5.4.4 always induces an orientation on the graph of the function with normal vector pointing in the positive z-direction.

(b) Find another parametrization of the graph of the function that induces the opposite orientation.

Solution. (a) Let $S \subset \mathbb{R}^3$ be the surface given by the graph $z = f(x, y)$ of a function f, for some region D in the (x, y)-plane. We parametrize it as $\alpha : D \to \mathbb{R}^3$ with

$$\alpha(u, v) = (u, v, f(u, v)),$$

where D is the same region D but in the (u, v)-plane. To calculate the normal vector, we calculate the tangent vectors:

$$\mathbf{T}_u = \left(1, 0, \frac{\partial f}{\partial u}\right), \qquad \mathbf{T}_v = \left(0, 1, \frac{\partial f}{\partial v}\right).$$

The normal vector is given by the cross-product:

$$\begin{aligned}\mathbf{n} =& \mathbf{T}_u \times \mathbf{T}_v \\ =& \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{pmatrix} \\ =& \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).\end{aligned}$$

In particular, we see that it always points in the positive z-direction, regardless of the details of the function f.

(b) We can do just as in the previous question; we can exchange u and v. We can write a new parametrization as $\alpha_2 : D_2 \to \mathbb{R}^3$, with D_2 being the same region as D but with u and v exchanged, and

$$\alpha(u, v) = (v, u, f(v, u)).$$

Then the tangent vectors are

$$\mathbf{T}_u = \left(0, 1, \frac{\partial f(v, u)}{\partial u}\right), \qquad \mathbf{T}_v = \left(1, 0, \frac{\partial f(v, u)}{\partial v}\right),$$

and the normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = \left(\frac{\partial f(v, u)}{\partial v}, \frac{\partial f(v, u)}{\partial u}, -1\right),$$

which is the same vector (with u and v exchanged) but with the opposite sign. This parametrization therefore induces the opposite orientation to the parametrization in (a).

4. Consider the surface S consisting of the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 5$, with normal vector pointing outside the cone. The boundary ∂S of S consists in two separate oriented closed curves. Realize these two curves as parametric curves, with orientations that are consistent with the induced orientation on the boundary ∂S.

Solution. The cone is shown in the figure below:

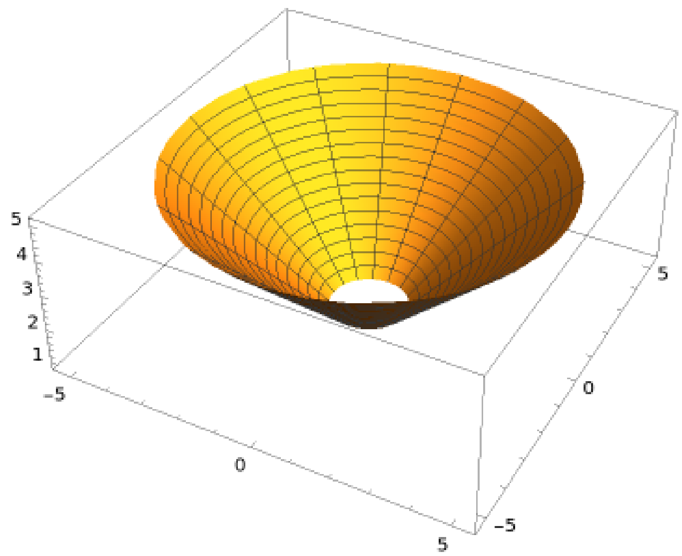


Figure 5.5.8 The part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 5$.

The boundary has two components: the top boundary circle and the bottom boundary circle. If we call C_1 the top boundary circle, and C_2 the bottom one, then $\partial S = C_1 \cup C_2$. These two curves are given by the curves:

$$C_1 =\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 25, z = 5\},$$
$$C_2 =\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 1\}.$$

What is the induced orientation on these boundary curves? The surface of the cone is oriented with an outwards pointing normal vector. To get the induced orientation on the boundary curves, we should walk along the curves, keeping the region on our left, with our head pointing in the direction of the normal vector. We see that C_1 should be oriented clockwise, while C_2 should be oriented counterclockwise. We now want to find parametrization of these two curves that are consistent with the induced orientations.

We start with C_1. This is a circle of radius 5 in the plane $z = 5$, oriented clockwise. We can parametrize the circle using polar coordinates, but we want to use sine for x and cosine for y so that the resulting curve is oriented clockwise. We get the parametrization $\alpha_1 : [0, 2\pi] \to \mathbb{R}^3$ with

$$\alpha_1(\theta) = (5\sin(\theta), 5\cos(\theta), 5).$$

As for C_2, it is the circle of radius 1 in the plane $z = 1$, oriented counterclockwise. We use polar coordinates again, but cosine for x and sine for y so that it is oriented counterclockwise. We get $\alpha_2 : [0, 2\pi] \to \mathbb{R}^3$ with:

$$\alpha_2(\theta) = (\cos(\theta), \sin(\theta), 1).$$

5.6 Surface integrals

Step 4: we define integration of a two-form along a parametric surface via pullback. We call those "surface integrals". Step 5: we show that the integral is well-defined: it is invariant under orientation-preserving reparametrizations, and changes sign under orientation-reversing reparametrizations. This ensures that the integral of the two-form is intrinsically defined in terms of the geometry of the image surface with its induced orientation. We also rewrite surface integrals in terms of vector fields.

Objectives

You should be able to:

- Define the surface integral of a two-form along a parametric surface in $\mathbb{R}^3$, and evaluate it.
- Rewrite the definition of surface integrals as flux integrals for the associated vector field.
- Show that surface integrals are invariant under orientation-preserving reparametrizations of the surface.
- Show that surface integrals change sign under reparametrizations of the surface that reverse its orientation.

5.6.1 The definition of surface integrals

We define the integral of a two-form along a parametric surface: we pull back to the region D and integrate.

Definition 5.6.1 Surface integrals. Let ω be a two-form on an open subset $U \subseteq \mathbb{R}^3$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface whose image surface $S = \alpha(D) \subset U$ is orientable. We define the **surface integral of ω along α** as:

$$\int_\alpha \omega = \int_D \alpha^* \omega,$$

where the integral on the right-hand-side is defined in Definition 5.3.1, with D given the canonical orientation (which is consistent with the induced orientation on S). $\Diamond$

What is neat is that by Definition 5.3.1, the integral on the right-hand-side can be written as a standard double integral from calculus. So using the pullback we reduced the evaluation of integrals of two-forms over surfaces to standard double integrals!

Example 5.6.2 An example of a surface integral. Consider the two-form $\omega = x\ dy \wedge dz + z\ dx \wedge dy$. Evaluate its surface integral along the oriented parametric surface $\alpha : D \to \mathbb{R}^3$ with $D = \{(u,v) \in \mathbb{R}^2 \mid u \in [0,1], v \in [0,2]\}$ and $\alpha(u,v) = (u, uv^2, v)$.

We calculate the pullback of the two-form:

$$\begin{aligned}\alpha^*\omega =& u\left(v^2\ du + 2uv\ dv\right) \wedge dv + v du \wedge \left(v^2\ du + 2uv\ dv\right)\\ =&(uv^2 + 2uv^2) du \wedge dv\\ =&3uv^2 du \wedge dv.\end{aligned}$$

The surface integral then becomes:

$$\int_\alpha \omega = \int_D 3uv^2\ du \wedge dv$$

$$
\begin{aligned}
&=3\int_0^2\int_0^1 uv^2\ dudv\\
&=3\int_0^2 v^2\left[\frac{u^2}{2}\right]_0^1 dv\\
&=\frac{3}{2}\int_0^2 v^2\ dv\\
&=\frac{v^3}{2}\Big|_0^2\\
&=4.
\end{aligned}
$$

Note that it is important here that we wrote the two-form $\alpha^*\omega$ in terms of the basic two-form $du \wedge dv$ (not $dv \wedge du$) before rewriting as a double integral, as the orientation on D is the canonical orientation with respect to the coordinates (u, v) on $D \subset \mathbb{R}^2$. □

5.6.2 Reparametrization-invariance and orientability of surface integrals

We defined surface integrals in terms of parametric surfaces: the parametrization was key, as we used it to pullback to D. But in the end, we would like our integral to be defined solely in terms of the image surface S itself with its orientation. If we use two different parametrizations that describe the same surface with the same induced orientation, we want the integral to be the same. The integral should not depend on how we describe the surface, as long as we preserve the induced orientation.

Lemma 5.6.3 Surface integrals are invariant under orientation-preserving reparametrizations. *Let ω be a two-form on an open subset $U \subseteq \mathbb{R}^3$. Let $\alpha : D_1 \to \mathbb{R}^3$ be a parametric surface whose image surface $S = \alpha(D_1) \subset U$ is orientable. Let $\phi : D_2 \to D_1$ be as in Lemma 5.5.7, so that $\phi^*\alpha : D_2 \to \mathbb{R}^3$ is another parametrization of the same image surface.*

1. *If ϕ is orientation-preserving (that is,* $\det J_\phi > 0$*), then*

$$\int_\alpha \omega = \int_{\phi^*\alpha} \omega.$$

2. *If ϕ is orientation-reversing (that is,* $\det J_\phi < 0$*), then*

$$\int_\alpha \omega = -\int_{\phi^*\alpha} \omega.$$

Proof. Let us rewrite the two surface integrals using the pullback. First,

$$\int_\alpha \omega = \int_{D_1} \alpha^*\omega.$$

Second,

$$\int_{\phi^*\alpha} \omega = \int_{D_2} (\alpha \circ \phi)^*\omega.$$

From Exercise 4.7.4.7, we know that we can pullback through a chain of maps $D_2 \xrightarrow{\phi} D_1 \xrightarrow{\alpha} \mathbb{R}^3$ in two different ways, but it gives the same thing: $(\alpha \circ \phi)^*\omega = \phi^*(\alpha^*\omega)$. Therefore,

$$\int_{\phi^*\alpha} \omega = \int_{D_2} \phi^*(\alpha^*\omega).$$

In the end, what we need to compare is two integrals over regions in $\mathbb{R}^2$. But this is precisely the result of Lemma 5.3.7, for the two-form $\alpha^*\omega$ on D_1. This lemma states that if $\det J_\phi > 0$, then

$$\int_{D_1} \alpha^*\omega = \int_{D_2} \phi^*(\alpha^*\omega),$$

while if $\det J_\phi < 0$,

$$\int_{D_1} \alpha^*\omega = -\int_{D_2} \phi^*(\alpha^*\omega),$$

We thus conclude that if the reparametrization is orientation-preserving,

$$\int_\alpha \omega = \int_{\phi^*\alpha} \omega,$$

while if it is orientation-reversing,

$$\int_\alpha \omega = -\int_{\phi^*\alpha} \omega.$$

■

Surface integrals are oriented and reparametrization-invariant, as we want. Nice! As a result, while we use a parametrization to define a surface integral, the integral can really be thought of as being defined intrinsically in terms of the surface S and its orientation.

5.6.3 Surface integrals in terms of vector fields

Let us now translate our definition in surface integrals in terms of vector fields, using the dictionary between differential forms and vector calculus concepts that we established.

We first need to do a bit of work to rephrase the pullback of a two-form along a parametric surface in terms of associated vector fields.

Lemma 5.6.4 The pullback of a two-form along a parametric surface in terms of vector fields. *Let $\omega = f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$ be a two-form on $U \subseteq \mathbb{R}^3$, with associated vector field $\mathbf{F} = (f, g, h)$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface whose image surface $S = \alpha(D) \subset U$ is orientable, with $\alpha(u, v) = (x(u, v), y(u, v), z(u, v))$. Let $\mathbf{T}_u, \mathbf{T}_v$ be the tangent vectors, and $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ be the normal vector. Then the pullback $\alpha^*\omega$ is:*

$$\alpha^*\omega = (\mathbf{F}(\alpha(u, v)) \cdot \mathbf{n})\, du \wedge dv.$$

Proof. This is just a calculation:

$$\begin{aligned}\alpha^*\omega =& f(\alpha(u,v)) \left(\frac{\partial y}{\partial u}\ du + \frac{\partial y}{\partial v}\ dv\right) \wedge \left(\frac{\partial z}{\partial u}\ du + \frac{\partial z}{\partial v}\ dv\right)\\ &+ g(\alpha(u,v)) \left(\frac{\partial z}{\partial u}\ du + \frac{\partial z}{\partial v}\ dv\right) \wedge \left(\frac{\partial x}{\partial u}\ du + \frac{\partial x}{\partial v}\ dv\right)\end{aligned}$$

$$
\begin{aligned}
&+ h(\alpha(u,v))\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) \wedge \left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right)\\
=&f(\alpha(u,v))\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) du \wedge dv\\
&+ g(\alpha(u,v))\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) du \wedge dv\\
&+ h(\alpha(u,v))\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv\\
=&\mathbf{F}(\alpha(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du \wedge dv.
\end{aligned}
$$

■

It is interesting to remark here that while pulling back a one-form along a parametric curve picked the tangential component of the vector field along the curve, pulling back a two-form along a parametric surface picks the normal component of the vector field. This has an interpretation in physics, as we will see. Just as line integrals were related to the calculation of work (and hence the tangential component of the force field to the direction of motion was the relevant one), surface integrals are related to the calculation of flux, for which the normal component of the force field is the relevant one.

With this result, we can rewrite surface integrals directly as double integrals:

Corollary 5.6.5 *Let ω be a two-form on an open subset $U \subseteq \mathbb{R}^3$, with associated vector field $\mathbf{F}$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface whose image surface $S = \alpha(D) \subset U$ is orientable, with $\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$. Then*

$$\int_\alpha \omega = \iint_D (\mathbf{F}(\alpha(u,v)) \cdot \mathbf{n})\, dA,$$

where on the right-hand-side this is a double integral over the region D in the uv-plane.

Proof. This follows directly from Lemma 5.6.4, the definition of surface integrals Definition 5.6.1, and the definition of integrals over region in $\mathbb{R}^2$ Definition 5.3.1. We have:

$$
\begin{aligned}
\int_\alpha \omega &= \int_D \alpha^*\omega\\
&= \int_D (\mathbf{F}(\alpha(u,v)) \cdot \mathbf{n})\, du \wedge dv\\
&= \iint_D (\mathbf{F}(\alpha(u,v)) \cdot \mathbf{n})\, dA.
\end{aligned}
$$

■

This is how surface integrals are generally defined in standard vector calculus textbooks.

Remark 5.6.6 Sometimes, the following shorthand notation is used:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_D (\mathbf{F}(\alpha(u,v)) \cdot \mathbf{n})\, dA.$$

Such integrals are also called **flux integrals**, because of the physics interpretation, which we will study in Section 5.9. The integral calculates the flux of the vector field $\mathbf{F}$ across the surface S in the direction of the normal vector $\mathbf{n}$ specified by the orientation.

Example 5.6.7 An example of a surface integral of a vector field. Calculate the surface integral of the vector field

$$\mathbf{F} = \frac{1}{x^2+y^2+z^2}(x, y, 0)$$

over the upper half-sphere of radius one, with the inwards orientation.

We can use our parametrization from Example 5.5.6 with $R = 1$, that is, $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(\theta, \phi) \in \mathbb{R}^2 \mid \theta \in \left[0, \frac{\pi}{2}\right], \phi \in [0, 2\pi]\},$$

and

$$\alpha(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)).$$

We showed in that example that the normal vector points outwards, so the orientation on the upper half-sphere induced by this parametrization is opposite to what is asked in the problem. Therefore, we will need to take minus the integral over the parametric surface α.

We calculated in Example 5.5.6 the normal vector:

$$\mathbf{n} = \left(\sin^2(\theta)\cos(\phi), \sin^2(\theta)\sin(\phi), \sin(\theta)\cos(\theta)\right).$$

The integrand is then (noting that $x^2 + y^2 + z^2 = 1$ on the surface):

$$\begin{aligned}
\mathbf{F}(\alpha(\theta, \phi)) \cdot \mathbf{n} &= (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), 0) \cdot \left(\sin^2(\theta)\cos(\phi), \sin^2(\theta)\sin(\phi), \sin(\theta)\cos(\theta)\right) \\
&= \sin^3(\theta)\cos^2(\phi) + \sin^3(\theta)\sin^2(\phi) \\
&= \sin^3(\theta).
\end{aligned}$$

The surface integral then becomes (recall that we need to add a minus sign since the surface integral is with respect to the inwards orientation, while our parametrization induces the outwards orientation):

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= -\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^3(\theta) d\theta d\phi \\
&= -\int_0^{2\pi} \left[-\cos(\theta) + \frac{1}{3}\cos^3(\theta)\right]_0^{\frac{\pi}{2}} d\phi \\
&= -\frac{2}{3}\int_0^{2\pi} d\phi \\
&= -\frac{4\pi}{3}.
\end{aligned}$$

□

5.6.4 Exercises

1. Find the surface integral of the two-form

$$\omega = y\ dy \wedge dz - (x+y)\ dz \wedge dx$$

along the lower half-sphere $x^2+y^2+z^2=4$, $z \leq 0$, with orientation given by an inwards pointing normal vector.

Solution. We use spherical coordinates to parametrize the lower half sphere of radius 2. That is, $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(\theta,\phi) \in \mathbb{R}^2 \mid \theta \in \left[\frac{\pi}{2}, \pi\right], \phi \in [0,2\pi]\},$$

and

$$\alpha(\theta,\phi) = (2\sin(\theta)\cos(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\theta)).$$

Here the inclination angle was restricted to be between $\pi/2$ and π, which amounts to considering the lower half-sphere with $z \leq 0$. We have already calculated in Example 5.5.6 the this parametrization induces a normal vector pointing outwards. The question is asking us to pick the opposite orientation, with normal vector pointing inwards. So we will have to add an overall minus sign to our surface integral.

To evaluate the surface integral we first need to calculate the pullback $\alpha^*\omega$. We get:

$$\begin{aligned}\alpha^*\omega =& 2\sin(\theta)\sin(\phi)(2\cos(\theta)\sin(\phi)d\theta + 2\sin(\theta)\cos(\phi)d\phi)) \wedge (-2\sin(\theta)d\theta)\\ &- (2\sin(\theta)\cos(\phi) + 2\sin(\theta)\sin(\phi))(-2\sin(\theta)d\theta) \wedge (2\cos(\theta)\cos(\phi)d\theta - 2\sin(\theta)\sin(\phi)d\phi)\\ =& -8\sin^3(\theta)\sin(\phi)\cos(\phi)d\phi \wedge d\theta - 8\sin^3(\theta)\sin(\phi)(\cos(\phi)+\sin(\phi))d\theta \wedge d\phi\\ =& -8\sin^3(\theta)\sin^2(\phi)d\theta \wedge d\phi.\end{aligned}$$

Then we integrate, recalling that we have to add a minus sign because of the orientation:

$$\begin{aligned}-\int_\alpha \omega =& -\int_D \alpha^*\omega\\ =& 8\int_{\pi/2}^{\pi}\int_0^{2\pi} \sin^3(\theta)\sin^2(\phi)d\phi d\theta\\ =& 8\pi\int_{\pi/2}^{\pi} \sin^3(\theta)d\theta\\ =& \frac{16\pi}{3},\end{aligned}$$

where I used the fact that

$$\int_0^{2\pi} \sin^2(\phi)d\phi = \pi, \qquad \int_{\pi/2}^{\pi} \sin^3(\theta)d\theta = \frac{2}{3},$$

which you can check independently using trigonometric identities to evaluate the definite integrals.

2. Find the flux (the surface integral) of the vector field

$$\mathbf{F}(x,y,z) = (ze^{xy}, -3ze^{xy}, 2xy)$$

along the parallelogram realized as the parametric surface $\alpha : D \to \mathbb{R}^3$ with $D = \{(u,v) \in \mathbb{R}^2 \mid u \in [0,2], v \in [0,1]\}$ and

$$\alpha(u,v) = (u+v, u-v, 1+2u+v).$$

Solution. Let us solve this problem using the vector field approach. We know the parametric surface; we need to calculate the normal vector. The tangent vectors are

$$\mathbf{T}_u = (1,1,2), \qquad \mathbf{T}_v = (1,-1,1).$$

(We note that those are constant vectors, since the image surface is planar.) The normal vector is then

$$\begin{aligned}\mathbf{n} =& \mathbf{T}_u \times \mathbf{T}_v \\ =& \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \\ =& (3,1,-2).\end{aligned}$$

We then calculate the surface integral:

$$\begin{aligned}\iint_D \mathbf{F} \cdot d\mathbf{S} =& \int_0^2 \int_0^1 ((1+2u+v)e^{u^2-v^2}, -3(1+2u+v)e^{u^2-v^2}, 2(u^2-v^2)) \cdot (3,1,-2)\ dvdu \\ =& \int_0^2 \int_0^1 \left(-4(u^2-v^2)\right)\ dvdu \\ =& -4 \int_0^2 \left[u^2 v - \frac{v^3}{3}\right]_{v=0}^{v=1} du \\ =& -4 \int_0^2 \left(u^2 - \frac{1}{3}\right)\ du \\ =& -4 \left[\frac{u^3}{3} - \frac{u}{3}\right]_{u=0}^{u=2} \\ =& -8.\end{aligned}$$

3. Find the surface integral of the two-form

$$\omega = x\ dy \wedge dz + z\ dx \wedge dy$$

over the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, with orientation given by a normal vector pointing outwards.

Solution. The cube is shown in the figure below. It has six sides, which we consider as separate parametric surfaces, S_i with $i = 1, \ldots, 6$.

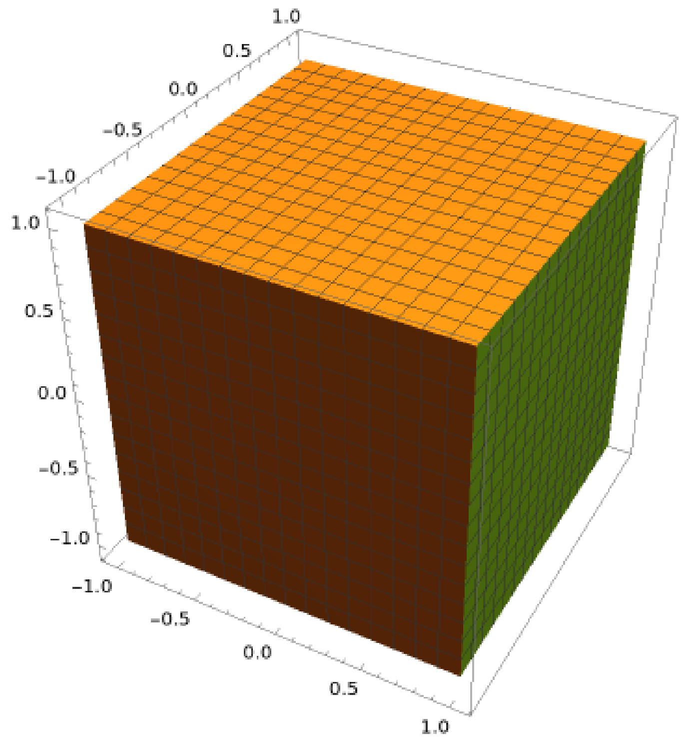


Figure 5.6.8 The surface S is the cube shown in the figure. We consider the six sides as separate parametric surfaces.

All six sides can be parametrized easily, with the same domain D. We write the parametrizations as $\alpha_i : D \to \mathbb{R}^3$, for $i = 1, \ldots, 6$, with

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [-1, 1], v \in [-1, 1]\},$$

and

$$\begin{aligned}\alpha_1(u, v) =&(1, u, v)\\ \alpha_2(u, v) =&(-1, u, v)\\ \alpha_3(u, v) =&(u, 1, v)\\ \alpha_4(u, v) =&(u, -1, v)\\ \alpha_5(u, v) =&(u, v, 1)\\ \alpha_6(u, v) =&(u, v, -1)\end{aligned}$$

Calculating the normal vectors, we see that the normal vectors for $\alpha_1, \alpha_4, \alpha_5$ point outwards, while the normal vectors for $\alpha_2, \alpha_3, \alpha_6$ point inwards. So we can write our desired surface integral as

$$\int_S \omega = \int_{S_1} \omega - \int_{S_2} \omega - \int_{S_3} \omega + \int_{S_4} \omega + \int_{S_5} \omega - \int_{S_6} \omega.$$

To calculate these surface integrals, we need the pullbacks. A straightforward calculation gives:

$$\alpha_1^* \omega = \alpha_5^* \omega = du \wedge dv,$$

$$\alpha_2^*\omega = \alpha_6^*\omega = -du \wedge dv,$$
$$\alpha_3^*\omega = \alpha_4^*\omega = 0.$$

Therefore,

$$\begin{aligned}\int_S \omega = &\int_{S_1} \omega - \int_{S_2} \omega - \int_{S_3} \omega + \int_{S_4} \omega + \int_{S_5} \omega - \int_{S_6} \omega \\ =&4\int_D du \wedge dv \\ =&4\int_{-1}^{1}\int_{-1}^{1} dudv \\ =&16.\end{aligned}$$

4. Let S be the boundary of the region in $\mathbb{R}^3$ enclosed by the cylinder $x^2 + y^2 = 4$, the paraboloid $z = x^2 + y^2$, and the plane $z = 9$. Find the surface integral of the vector field

$$\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$$

along S with orientation given by a normal vector pointing inwards.

Solution. We see that the cylinder $x^2 + y^2 = 4$ and the paraboloid $z = x^2 + y^2$ intersect along the circle $x^2 + y^2 = 4$ in the plane $z = 4$. Thus the surface S has three components: S_1 is the part the paraboloid $z = x^2 + y^2$ with $0 \leq z \leq 4$; S_2 is the part of the cylinder $x^2 + y^2 = 4$ with $4 \leq z \leq 9$; and S_3 is the disk $x^2 + y^2 \leq 4$ in the plane $z = 9$. This is shown in Figure 5.6.9.

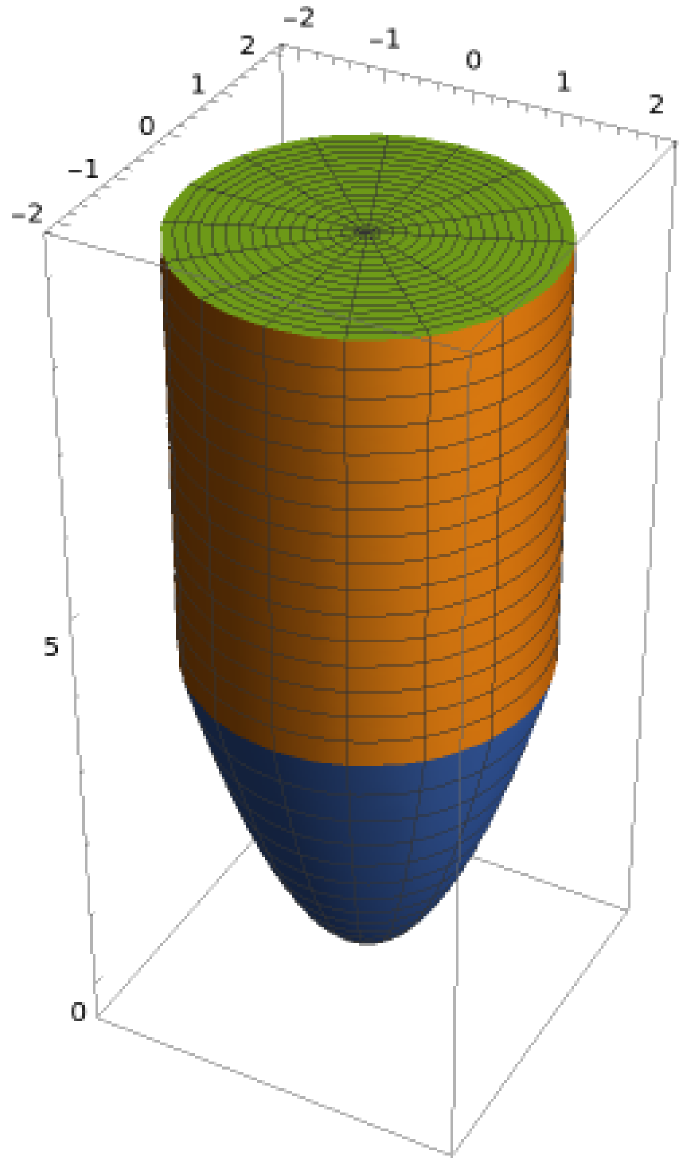


Figure 5.6.9 The surface S has three components: S_1 is the part of the paraboloid (in blue) with $z \leq 4$; S_2 is the lateral surface of the cylinder (in orange) for $4 \leq z \leq 9$; S_3 is the disk $x^2 + y^2 \leq 4$ in the plane $z = 9$ (in green).

To evaluate the surface integral of ω over S, we evaluate it over the three components separately and then add up the integrals.

Let us consider S_1 first, which is the part of the paraboloid $z = x^2 + y^2$ with $0 \leq z \leq 4$. We realize it as the parametric surface $\alpha_1 : D_1 \to \mathbb{R}^3$ with

$$D_1 = \{(u, \theta) \in \mathbb{R}^2 \mid u \in [0, 2], \theta \in [0, 2\pi]\}$$

and

$$\alpha_1(u, \theta) = (u\cos(\theta), u\sin(\theta), u^2).$$

We need to make sure that the normal vector points inwards. The tangent vectors are

$$\mathbf{T}_u = (\cos(\theta), \sin(\theta), 2u), \qquad \mathbf{T}_\theta = (-u\sin(\theta), u\cos(\theta), 0).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_\theta = (-2u^2\cos(\theta), -2u^2\sin(\theta), u).$$

As $u \geq 0$, we see that the normal vector points in the positive z-direction, which means that it points inwards. So we're good!

We then evaluate the surface integral:

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}_1 =& \int_0^2\int_0^{2\pi} \Big((u^2\cos^2(\theta), u^2\sin^2(\theta), u^4)\cdot(-2u^2\cos(\theta), -2u^2\sin(\theta), u)\Big)\ d\theta du\\ =& \int_0^2\int_0^{2\pi}\Big(-2u^4(\cos^3(\theta)+\sin^3(\theta)) + u^5\Big)\,d\theta du\\ =& 2\pi\int_0^2 u^5\ du\\ =& \frac{64\pi}{3},\end{aligned}$$

where we used the fact that $\int_0^{2\pi}\cos^3(\theta)d\theta = \int_0^{2\pi}\sin^3(\theta)d\theta = 0$.

Let us now consider S_2, which is the cylinder $x^2+y^2=4$, $4\leq z\leq 9$. We parametrize it as $\alpha_2 : D_2 \to \mathbb{R}^3$ with

$$D_2 = \{(u,\theta)\in\mathbb{R}^2 \mid u\in[4,9], \theta\in[0,2\pi]\}$$

and

$$\alpha_2(u,\theta) = (2\cos(\theta), 2\sin(\theta), u).$$

We look at the normal vector. The tangent vectors are

$$\mathbf{T}_u = (0,0,1), \qquad \mathbf{T}_\theta = (-2\sin(\theta), 2\cos(\theta), 0),$$

and the normal vector is

$$\mathbf{n} = \mathbf{T}_u\times\mathbf{T}_\theta = (-2\cos(\theta), -2\sin(\theta), 0).$$

We thus see that at a point with coordinates $(2\cos(\theta), 2\sin(\theta), u)$ on the cylinder, the normal vector is $(-2\cos(\theta), -2\sin(\theta), 0)$, which points inwards. Good!

We evaluate the surface integral:

$$\begin{aligned}\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}_2 =& \int_4^9\int_0^{2\pi}\Big((4\cos^2(\theta), 4\sin^2(\theta), u^2)\cdot(-2\cos(\theta), -2\sin(\theta), 0)\Big)\,d\theta du\\ =& \int_4^9\int_0^{2\pi}\Big(-8\cos^3(\theta) - 8\sin^3(\theta)\Big)\,d\theta du\\ =& 0,\end{aligned}$$

since, as above, $\int_0^{2\pi}\cos^3(\theta)d\theta = \int_0^{2\pi}\sin^3(\theta)d\theta = 0$.

Finally, we consider S_3, which is the disk $x^2 + y^2 \leq 4$ in the plane $z = 9$. We parametrize it as $\alpha_3 : D_3 \to \mathbb{R}^3$ with

$$D_3 = \{(u, \theta) \in \mathbb{R}^2 \mid u \in [0, 2], \theta \in [0, 2\pi]\}$$

and

$$\alpha_3(u, \theta) = (u \sin(\theta), u \cos(\theta), 9).$$

The tangent vectors are

$$\mathbf{T}_u = (\sin(\theta), \cos(\theta), 0), \qquad \mathbf{T}_\theta = (u \cos(\theta), -u \sin(\theta), 0),$$

and the normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_\theta = (0, 0, -u).$$

As $u \geq 0$, this points in the negative z-direction, that is, it points inwards, as we want.

We evaluate the surface integral:

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot d\mathbf{S}_3 =& \int_0^2 \int_0^{2\pi} \Big((u^2 \sin^2(\theta), u^2 \cos^2(\theta), 81) \cdot (0, 0, -u) \Big) \, d\theta du \\ =& \int_0^2 \int_0^{2\pi} (-81u) d\theta du \\ =& - 162\pi \int_0^2 u du, \\ =& - 324\pi.\end{aligned}$$

Finally, we compute the surface integral over S by adding up the three surface integrals aboves. We get:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} =& \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}_1 + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}_2 + \iint_{S_3} \mathbf{F} \cdot d\mathbf{S}_3 \\ =& \frac{64\pi}{3} + 0 - 324\pi \\ =& - \frac{908\pi}{3}.\end{aligned}$$

Done! :-)

5. An "inverse square field" is a vector field $\mathbf{F}$ that is inversely proportional to the square of the distance from the origin. It is very important in physics, as it describes many physical phenomena, such a gravity, electrostatics, etc. It can be written as

$$\mathbf{F}(x, y, z) = C \frac{1}{(x^2 + y^2 + z^2)^{3/2}} (x, y, z),$$

for some constant $C \in \mathbb{R}$. Show that the flux (the surface integral) of $\mathbf{F}$ across a sphere S centered at the origin (in the outwards direction) is independent of the radius of S. This is one of the very nice properties of inverse square fields!

Solution. As usual we use spherical coordinates to parametrize the sphere of fixed radius R, as $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(\theta, \phi) \in \mathbb{R}^2 \mid \theta \in [0, \pi], \phi \in [0, 2\pi]\},$$

and

$$\alpha(\theta, \phi) = (R \sin(\theta) \cos(\phi), R \sin(\theta) \sin(\phi), R \cos(\theta)).$$

Let use differential forms to calculate the surface integral. To the vector field $\mathbf{F}$ we associated the two-form

$$\omega = C \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \left(x \ dy \wedge dz + y \ dz \wedge dx + z \ dx \wedge dy \right).$$

To calculate the surface integral, we calculate the pullback, noting that $x(\theta, \phi)^2 + y(\theta, \phi)^2 + z(\theta, \phi)^2 = R^2$:

$$\begin{aligned}
\alpha^* \omega =& \frac{C}{R^3} (R \sin(\theta) \cos(\phi) (R \cos(\theta) \sin(\phi) d\theta + R \sin(\theta) \cos(\phi) d\phi) \wedge (-R \sin(\theta) d\theta) \\
& + R \sin(\theta) \sin(\phi) (-R \sin(\theta) d\theta) \wedge (R \cos(\theta) \cos(\phi) d\theta - R \sin(\theta) \sin(\phi) d\phi) \\
& + R \cos(\theta) (R \cos(\theta) \cos(\phi) d\theta) - R \sin(\theta) \sin(\phi) d\phi) \wedge (R \cos(\theta) \sin(\phi) d\theta + R \sin(\theta) \cos(\phi) d\phi)) \\
=& \frac{CR^3}{R^3} (\sin^3(\theta) \cos^2(\phi) + \sin^3(\theta) \sin^2(\phi) + \cos^2(\theta) \sin(\theta) \cos^2(\phi) + \cos^2(\theta) \sin(\theta) \sin^2(\phi)) d\theta \wedge d\phi \\
=& C \sin(\theta) d\theta \wedge d\phi.
\end{aligned}$$

In particular, we see that it does not depend on the radius R of the sphere! The surface integral is then

$$\begin{aligned}
\int_\alpha \omega =& \int_D \alpha^* \omega \\
=& C \int_0^{2\pi} \int_0^\pi \sin(\theta) d\theta d\phi \\
=& 2C \int_0^{2\pi} d\phi \\
=& 4\pi C,
\end{aligned}$$

which is indeed independent of the radius R.

5.7 Green's theorem

Step 6: we study what happens if we integrate an exact two-form. We start by studying integrals of exact two-forms over bounded regions in $\mathbb{R}^2$: this gives rise to Green's theorem.

Objectives

You should be able to:

- State Green's theorem for integrals of exact two-forms over closed bounded regions in $\mathbb{R}^2$.

- Use Green's theorem to evaluate integrals of exact two-forms over closed bounded regions in $\mathbb{R}^2$.
- Use Green's theorem to evaluate line integrals of one-forms along simple closed curves in $\mathbb{R}^2$.
- Rephrase Green's theorem in terms of the associated vector fields.

5.7.1 Green's theorem

Let us start by recalling the corresponding statement for integrals of exact one-forms over intervals in $\mathbb{R}$. Let f be a zero-form on $U \subset \mathbb{R}$. Let $[a,b] \in U$ with the canonical orientation of increasing numbers. The boundary of the interval, with its induced orientation, is $\partial[a,b] = \{(b,+),(a,-)\}$. Then the integral of the exact one-form $\omega = df$ along $[a,b]$ can be evaluated as:

$$\int_{[a,b]} df = \int_{\partial[a,b]} f = f(b) - f(a).$$

This is nothing else but the Fundamental Theorem of Calculus, rewritten in a fancy way using one-forms and zero-forms.

We now want to do something similar for exact two-forms over closed bounded regions in $\mathbb{R}^2$.

Theorem 5.7.1 Green's theorem. *Let η be a one-form on an open subset $U \subseteq \mathbb{R}^2$. Let $D \subset U$ be a closed bounded region with canonical orientation, and let ∂D be its boundary with the induced orientation as in Definition 5.2.9. Then the integral of the exact two-form $d\eta$ along D can be evaluated as:*

$$\int_D d\eta = \int_{\partial D} \eta,$$

where on the right-hand-side this is understood as the integral of the one-form η over the boundary curve $\partial D \subset \mathbb{R}^2$, which is realized as a parametric curve (or union of parametric curves) compatible with the induced orientation.

We can be a little more explicit. If we write $\eta = f\ dx + g\ dy$, then $d\eta = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$. The statement of Green's theorem becomes

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_{\partial D} (f\ dx + g\ dy)\,.$$

Proof. We will only prove Green's theorem for recursively supported regions. (In fact, we will only write the proof for x-supported regions, but the proof for y-supported regions is analogous.) For more general closed bounded regions, one can prove Green's theorem by rewriting the region as an union of recursively supported regions.

We write $\eta = f\ dx + g\ dy$ and $d\eta = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$.

Let D be an x-supported region of the form:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], u(x) \leq y \leq v(x)\},$$

for some continuous functions $u, v : \mathbb{R} \to \mathbb{R}$. The boundary ∂D is a closed simple curve, with counterclockwise orientation. It can be split into four curves:

1. C_1 is the vertical line $x = b$ between $y = u(b)$ and $y = v(b)$;
2. C_2 is the curve $y = v(x)$ between $x = b$ and $x = a$;
3. C_3 is the vertical line $x = a$ between $y = v(a)$ and $y = u(a)$;
4. C_4 is the curve $y = u(x)$ between $x = a$ and $x = b$.

In fact, for C_2 and C_3, we will pick the opposite orientations (as it simplifies the parametrizations), and add a negative sign in front of the line integrals. Those four curves can be realized as parametric curves (C_2 and C_3 have opposite orientations):

1. $\alpha_1 : [0, 1] \to \mathbb{R}^2$ with $\alpha_1(t) = (b, u(b) + (v(b) - u(b))t)$;
2. $\alpha_2 : [a, b] \to \mathbb{R}^2$ with $\alpha_2(t) = (t, v(t))$;
3. $\alpha_3 : [0, 1] \to \mathbb{R}^2$ with $\alpha_3(t) = (a, u(a) + (v(a) - u(a))t)$;
4. $\alpha_4 : [a, b] \to \mathbb{R}^2$ with $\alpha_4(t) = (t, u(t))$;

Calculating the pullbacks, we evaluate the line integrals:

$$\begin{aligned}
\int_{\alpha_1} \eta &= \int_0^1 g(b, u(b) + (v(b) - u(b))t)(v(b) - u(b))\ dt, \\
\int_{\alpha_2} \eta &= \int_a^b \left(f(t, v(t)) + g(t, v(t))v'(t)\right)\ dt, \\
\int_{\alpha_3} \eta &= \int_0^1 g(a, u(a) + (v(a) - u(a))t)(v(a) - u(a))\ dt, \\
\int_{\alpha_4} \eta &= \int_a^b \left(f(t, u(t)) + g(t, u(t))u'(t)\right)\ dt.
\end{aligned}$$

Putting those together, remembering that we need to add a negative sign for C_2 and C_3, we get:

$$\begin{aligned}
\int_{\partial D} \eta &= \int_{\alpha_1} \eta - \int_{\alpha_2} \eta - \int_{\alpha_3} \eta + \int_{\alpha_4} \eta \\
&= \int_0^1 \left(g(b, u(b) + (v(b) - u(b))t)(v(b) - u(b)) - g(a, u(a) + (v(a) - u(a))t)(v(a) - u(a))\right) dt \\
&\quad + \int_a^b \left(f(t, u(t)) - f(t, v(t)) + g(t, u(t))u'(t) - g(t, v(t))v'(t)\right)\ dt.
\end{aligned}$$

Next, we need to evaluate the integral $\int_D d\eta$. Here we will do our favourite trick: the pullback. Instead of using the x-supported domain D directly, we will pullback to a rectangular domain. This will allow us to integrate the two-form. Consider the function $\phi : D_2 \to D$, where

$$D_2 = \{(s, t) \in \mathbb{R}^2 \mid s \in [a, b], t \in [0, 1]\},$$

with

$$\phi(s, t) = (s, u(s) + (v(s) - u(s))t).$$

We see that $\phi(D_2) = D$, and that the function is bijective. So by Lemma 5.3.7, we know that

$$\int_{D_2} \phi^*(d\eta) = \int_{D_1} d\eta,$$

so we may as well calculate the left-hand-side. In fact, we also know that $\phi^*(d\eta) = d(\phi^*\eta)$. So let us calculate this pullback. First,

$$\phi^*\eta = f(\phi(s,t))\ ds + g(\phi(s,t))(u'(s) + (v'(s) - u'(s))t)\ ds + g(\phi(s,t))(v(s) - u(s))\ dt.$$

Then

$$\begin{aligned} d(\phi^*\eta) = \Big(& -\frac{\partial f(\phi(s,t))}{\partial t} - \frac{\partial}{\partial t}\left(g(\phi(s,t))(u'(s) + (v'(s) - u'(s))t)\right) \\ & + \frac{\partial}{\partial s}\left(g(\phi(s,t))(v(s) - u(s))\right)\Big)\, ds \wedge dt. \end{aligned}$$

Thus

$$\begin{aligned} \int_{D_2} \phi^*(d\eta) = & \int_a^b \int_0^1 \left(-\frac{\partial f(\phi(s,t))}{\partial t} - \frac{\partial}{\partial t}\left(g(\phi(s,t))(u'(s) + (v'(s) - u'(s))t)\right)\right) dtds \\ & + \int_0^1 \int_a^b \left(\frac{\partial}{\partial s}\left(g(\phi(s,t))(v(s) - u(s))\right)\right) dsdt, \end{aligned}$$

where we used Fubini's theorem to exchange the order of two integrals in the second line. We can then evaluate the inner integrals using the Fundamental Theorem of Calculus, since they are definite integrals of derivatives. We get:

$$\begin{aligned} \int_{D_2} \phi^*(d\eta) = & \int_a^b \left(f(\phi(s,0)) - f(\phi(s,1)) + g(\phi(s,0))u'(s) - g(\phi(s,1))v'(s)\right) ds \\ & + \int_0^1 \left(g(\phi(b,t))(v(b) - u(b)) - g(\phi(a,t))(v(a) - u(a))\right)\ dt. \end{aligned}$$

Substituting back the expression for $\phi(s,t)$, we get:

$$\begin{aligned} \int_{D_2} \phi^*(d\eta) = & \int_a^b \left(f(s,u(s)) - f(s,v(s)) + g(s,u(s))u'(s) - g(s,v(s))v'(s)\right)\ ds \\ & + \int_0^1 \left(g(b, u(b) + (v(b) - u(b))t)(v(b) - u(b)) - g(a, u(a) + (v(a) - u(a))t)(v(a) - u(a))\right)\ dt. \end{aligned}$$

Magic: this is exactly the same as the expression that we found many lines above for the line integral $\int_{\partial D} \eta$! Woot woot! Therefore

$$\int_D d\eta = \int_{\partial D} \eta.$$

■

Remark 5.7.2 If D is simply connected, then ∂D is a simple closed curve, and D is the closed curve with its interior. In this case the line integral $\int_{\partial D} \eta$ is simply the line integral of the one-form η along the simple closed curve with counterclockwise orientation.

However, in the statement of Green's theorem, D can be more general. For instance, it could be an annulus. Then ∂D would have two components: the inner and outer circle. The line integral $\int_{\partial D} \eta$ would then be the sum of the line integrals of the one-form η along the two circles; the outer circle with counterclockwise orientation, and the inner circle with clockwise orientation, as this is the induced orientation on the boundary according to Definition 5.2.9.

Remark 5.7.3 Following up on the previous remark, we note that we can read Green's theorem in two different ways, which is generally the case for all integral theorems of vector calculus. We could say:

1. The integral of the exact two-form $d\eta$ over the region $D \subset \mathbb{R}^2$ with canonical orientation is equal to the integral of the one-form η over the boundary ∂D with the induced orientation.

2. The integral of the one-form η over the simple closed curve $\partial D \subset \mathbb{R}^2$ with canonical orientation is equal to the integral of its exterior derivative $d\eta$ over the interior D with canonical orientation.

This is just two different readings of the same statement, depending on whether you start on the left-hand-side or the right-hand-side. Consequently, we can use Green's theorem either to evaluate integrals of exact two-forms via reading 1, or to evaluate line integrals over simple closed curves via reading 2.

In practice however, Green's theorem is generally useful mostly to evaluate line integrals by transforming them into surface integrals.

Example 5.7.4 Using Green's theorem to calculate line integrals. Find the line integral of the one-form $\omega = xy\ dx + (x + y)\ dy$ over the rectangle with vertices $(-2, -1), (2, -1), (2, 0), (-2, 0)$, with a clockwise orientation.

We could evaluate the line integral using previous techniques, by rewriting the curve as four parametric curves for each line segment and then use the definition of line integrals. But let us instead use Green's theorem.

Let us denote the rectangular curve by C. It is the boundary of the region D consisting of the interior of the rectangle with its boundary:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [-2, 2], y \in [-1, 0]\}.$$

Thus we can use Green's theorem to rewrite the line integral along C as a surface integral over D.

We have to be careful with orientation though. If D has canonical orientation, the induced orientation on the boundary rectangle C will be counterclockwise. But the question is asking us to evaluate the line integral along C with clockwise orientation. So in Green's theorem, if we relate the line integral to the surface integral over D with canonical orientation, we will need to introduce a minus sign. More precisely, let us denote C_- to be the rectangle with clockwise orientation, C_+ the rectangle with counterclockwise orientation, and D the rectangular region with canonical (counterclockwise) orientation. Green's theorem states:

$$\int_{C_-} \omega = -\int_{C_+} \omega = -\int_D d\omega.$$

So instead of calculating the line integral in the problem, we can calculate $-\int_D d\omega$.

The two-form $d\omega$ is:

$$d\omega = x dy \wedge dx + dx \wedge dy = (1-x) dx \wedge dy.$$

The integral can be evaluated:

$$\begin{aligned}
\int_D d\omega &= \int_{-1}^{0} \int_{-2}^{2} (1-x)\ dxdy \\
&= \int_{-1}^{0} \left[x - \frac{x^2}{2} \right]_{-2}^{2}\ dy \\
&= \int_{-1}^{0} (2-(-2)-2+2)\ dy \\
&= 4(0-(-1)) \\
&= 4.
\end{aligned}$$

Therefore, the line integral of ω along C_- is

$$\int_{C_-} \omega = -\int_D d\omega = -4.$$

It is a good exercise to check that this is the correct answer by evaluating the line integral using the standard approach with parametric curves. □

While Green's theorem is mostly useful to evaluate line integrals, we can also use Green's theorem in the reverse direction, to evaluate integrals of exact two-forms. Here is an example.

Example 5.7.5 Area of an ellipse. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The area is given by integrating the basic two-form $\omega = dx \wedge dy$ over the region D bounded by the ellipse with canonical orientation. We could calculate this integral directly, as a double integral. Or we can use Green's theorem, if ω is exact. But it certainly is: for instance, we can write $\omega = d\eta$ with $\eta = \frac{1}{2}(x\ dy - y\ dx)$ (we could choose other one-forms such that $d\eta = \omega$ as well, but this one gives a nice and easy integral). Then Green's theorem states:

$$\int_D \omega = \int_C \eta,$$

where on the right-hand-side we are evaluating the line integral of $\eta = \frac{1}{2}(x\ dy - y\ dx)$ along the ellipse with counterclockwise orientation.

We can parametrize the ellipse as $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with $\alpha(t) = (a\cos(t), b\sin(t))$. The orientation is counterclockwise as required. The pullback $\alpha^*\eta$ is:

$$\alpha^*\eta = \left(\frac{ab}{2}\cos^2(t) - \frac{ab}{2}\sin(t)(-\sin(t)) \right)\ dt = \frac{ab}{2} dt.$$

The line integral is then

$$\int_C \eta = \int_0^{2\pi} \frac{ab}{2}\ dt$$

$$= \pi ab,$$

which is the area of the ellipse. □

5.7.2 Vector form of Green's theorem

As always, we can translate our results to vector calculus concepts using our dictionary. It is a bit artificial though here, since Green's theorem really lives in two dimensions, while our vector calculus concepts live in three dimensions. But the idea is to think of the region D as being the trivial parametric surface in $\mathbb{R}^3$ where we embed it trivially in the xy-plane. That is, $\alpha : D \to \mathbb{R}^3$, with $\alpha(x, y) = (x, y, 0)$. The tangent vectors are $\mathbf{T}_x = (1, 0, 0)$ and $\mathbf{T}_y = (0, 1, 0)$, and the normal vector is of course $\mathbf{n} = (0, 0, 1) = \mathbf{e}_3$.

To the one-form $\eta = f\ dx + g\ dy$, we associate the vector field $\mathbf{F} = (f, g, 0)$. To the two-form $\omega = d\eta$ is then associated the vector field $\boldsymbol{\nabla} \times \mathbf{F}$. Using Corollary 5.6.5, we then see that we can rewrite the surface integral as

$$\int_D d\eta = \iint_D \left((\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{e}_3 \right)\ dA.$$

In other words, the integrand is the z-component of the curl $\boldsymbol{\nabla} \times \mathbf{F}$.

As for the other side of Green's theorem, it is a line integral of the one-form η over the curve ∂D. Assuming that we parametrize the curve with a position function $\mathbf{r}$, we can rewrite the line integral as

$$\int_{\partial D} \eta = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

The result is the vector form of Green's theorem:

$$\iint_D \left((\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{e}_3 \right)\ dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

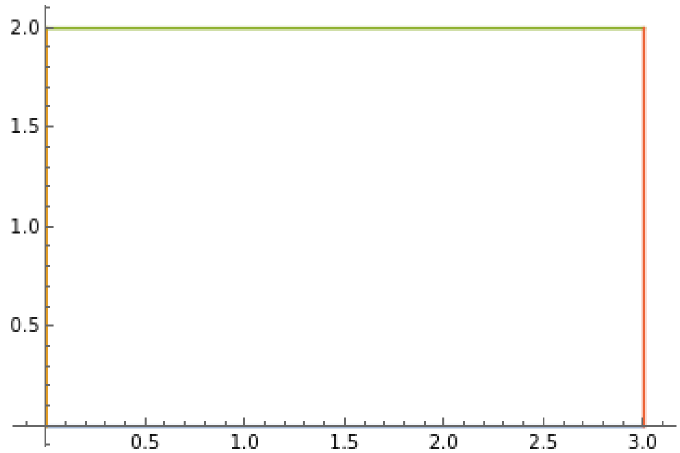
5.7.3 Exercises

1. Evaluate the line integral of the one-form

$$\omega = e^y\ dx + e^x\ dy$$

along the rectangle with vertices $(0, 0), (3, 0), (3, 2), (0, 2)$, counterclockwise, using two different methods: (a) directly, (b) using Green's theorem.

Solution. (a) Let us call ∂D the path along the rectangle counterclockwise. It is shown



in the figure below. **Figure 5.7.6** The path ∂D along the rectangle.

To evaluate the line integral directly, we need to parametrize separately the four line segments. We write the parametrizations as $\alpha_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, 4$, with

$$\begin{aligned}\alpha_1(t) =&(3t, 0),\\ \alpha_2(t) =&(3, 2t),\\ \alpha_3(t) =&(3-3t, 2),\\ \alpha_4(t) =&(0, 2-2t).\end{aligned}$$

We calculate the pullbacks:

$$\begin{aligned}\alpha_1^*\omega =&3\ dt,\\ \alpha_2^*\omega =&2e^3\ dt,\\ \alpha_3^*\omega =&-3e^2\ dt,\\ \alpha_4^*\omega =&-2\ dt.\end{aligned}$$

Putting all this together, the line integral becomes

$$\begin{aligned}\int_{\partial D}\omega =& \int_{\alpha_1}\omega + \int_{\alpha_2}\omega + \int_{\alpha_3}\omega + \int_{\alpha_4}\omega\\ =&\int_0^1 \left(3 + 2e^3 - 3e^2 - 2\right)\ dt\\ =&1 + 2e^3 - 3e^2.\end{aligned}$$

(b) We now use Green's theorem to calculate the line integral. By Green's theorem, we know that

$$\int_{\partial D}\omega = \int_D d\omega,$$

where D is the interior of the rectangle with its boundary, that is,

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,3], y \in [0,2]\},$$

with canonical orientation. We calculate the exterior derivative:

$$\begin{aligned} d\omega =& e^y \ dy \wedge dx + e^x \ dx \wedge dy \\ =& (e^x - e^y) \ dx \wedge dy. \end{aligned}$$

The integral of the two-form is:

$$\begin{aligned} \int_D d\omega =& \int_D (e^x - e^y) \ dx \wedge dy \\ =& \int_0^2 \int_0^3 (e^x - e^y) dx dy \\ =& \int_0^2 \left(e^3 - 1 - 3e^y\right) \ dy \\ =& 2e^3 - 2 - 3e^2 + 3 \\ =& 1 + 2e^3 - 3e^2. \end{aligned}$$

Therefore,

$$\int_{\partial D} \omega = 1 + 2e^3 - 3e^2,$$

which is the same answer as in part (a), as it should.

2. Use Green's theorem to evaluate the line integral of the vector field $\mathbf{F}(x, y) = (x^{1/3} + y, x^{4/3} - y^{1/3})$ along the triangle with vertices $(1, 3), (2, 2), (1, 1)$, clockwise.

Solution. Let us call ∂D the path around the triangle clockwise. It is shown in the figure below.

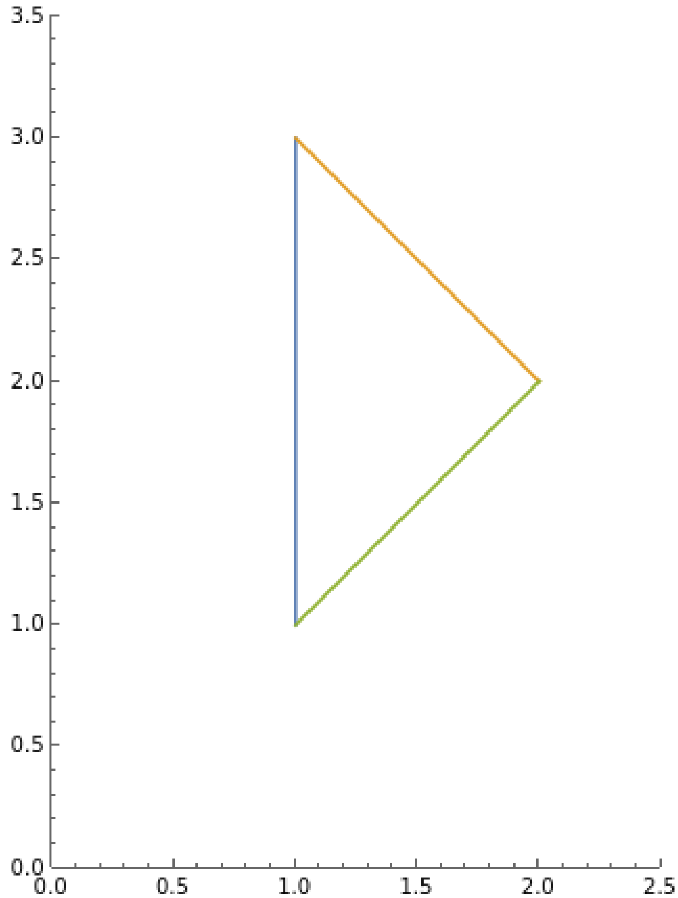


Figure 5.7.7 The path ∂D along the triangle.

We associate to the vector field $\mathbf{F}$ the one-form $\omega = (x^{1/3} + y)\ dx + (x^{4/3} - y^{1/3})\ dy$. Since the path ∂D is oriented clockwise (negative orientation), by Green's theorem we know that

$$\int_{\partial D} \omega = -\int_D d\omega,$$

where D is the interior of the triangle with its boundary, with canonical orientation. We can write equations for the three sides of the triangle. The blue line has equation $x = 1$; the green line, $y = x$; the orange line, $y = -x + 4$. Using these equations, we can describe D as an x-supported region:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [1, 2], x \leq y \leq -x + 4\}.$$

We calculate the exterior derivative $d\omega$:

$$\begin{aligned} d\omega =& dy \wedge dx + \frac{4}{3} x^{1/3} dx \wedge dy \\ =& \left(\frac{4}{3} x^{1/3} - 1 \right) dx \wedge dy. \end{aligned}$$

Its integral along D is:

$$\int_D d\omega = \int_1^2 \int_x^{-x+4} \left(\frac{4}{3} x^{1/3} - 1 \right)\ dy dx$$

$$
\begin{aligned}
&= \int_1^2 \left(\frac{4}{3}x^{1/3} - 1\right)\Big[y\Big]_x^{-x+4}\, dx \\
&= \int_1^2 \left(\frac{4}{3}x^{1/3} - 1\right)(-x + 4 - x)\, dx \\
&= \int_1^2 \left(-\frac{8}{3}x^{4/3} + 2x + \frac{16}{3}x^{1/3} - 4\right)\, dx \\
&= \left(-\frac{8}{7}x^{7/3} + x^2 + 4x^{4/3} - 4x\right)_{x=1}^{x=2} \\
&= -\frac{32}{7}2^{1/3} + 4 + 8 \cdot 2^{1/3} - 8 + \frac{8}{7} - 1 - 4 + 4 \\
&= \frac{3}{7}\left(8 \cdot 2^{1/3} - 9\right).
\end{aligned}
$$

Therefore,

$$\int_{\partial D} \omega = -\int_D d\omega = -\frac{3}{7}\left(8 \cdot 2^{1/3} - 9\right).$$

3. Use Green's theorem to find the work done by the force

$$\mathbf{F}(x, y) = (x^2 + y^2, xy)$$

while moving an object first along the parabola $y = x^2$ from the origin to the point $(2, 4)$, then along a horizontal line back to the y-axis, and then back to the origin along a vertical line.

Solution. We denote the closed path by ∂D. It is shown in the figure below.

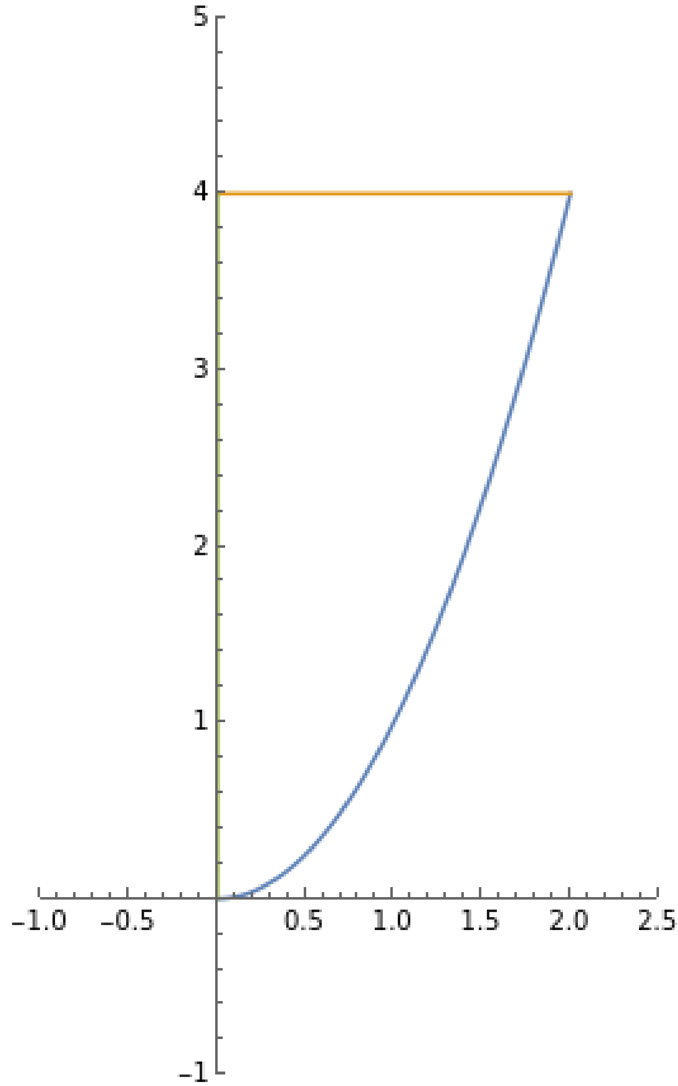


Figure 5.7.8 The closed path ∂D.

From the description of the path we know that the object is moving counterclockwise along the path. Therefore, if we associate the one-form $\omega = (x^2 + y^2)\ dx + xy\ dy$ to the vector field $\mathbf{F}$, by Green's theorem we know that

$$\int_{\partial D} \omega = \int_D d\omega,$$

with D being the region enclosed by the path (with its boundary), with canonical orientation. The region D can be described as an x-supported region:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 2], x^2 \leq y \leq 4\}.$$

The exterior derivative $d\omega$ is:

$$d\omega = 2y\ dy \wedge dx + y\ dx \wedge dy$$

$$= -\, y \; dx \wedge dy$$

Its integral along D is:

$$\begin{aligned}
\int_D d\omega &= -\int_D y \; dx \wedge dy \\
&= -\int_0^2 \int_{x^2}^4 y \; dy dx \\
&= -\int_0^2 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=4} \; dx \\
&= -\int_0^2 \left(8 - \frac{x^4}{2} \right) \; dx \\
&= -\left(16 - \frac{16}{5} \right) \\
&= -\,\frac{64}{5}.
\end{aligned}$$

Therefore, by Green's theorem, the work done by the force **F** while moving an object along the path ∂D is:

$$\int_{\partial D} \omega = -\frac{64}{5}.$$

4. Suppose that a polygon has vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, in counterclockwise order. The area A of the polygon is given by integrating the basic two-form $\omega = dx \wedge dy$ over the polygon. Use Green's theorem to show that

$$A = \frac{1}{2} \left((x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \ldots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n) \right).$$

This is a well known formula for the area of an arbitrary polygon, see for instance https://en.wikipedia.org/wiki/Polygon#Area.

Solution. We denote by ∂D the closed path that goes around the polygon counterclockwise. It consists of n line segments L_i, $i = 1, \ldots, n$, with L_i being the line segment between the points (x_i, y_i) and (x_{i+1}, y_{i+1}) (for simplicity of notation, we define $(x_{n+1}, y_{n+1}) := (x_1, y_1)$).

Further, we notice that the two $\omega = dx \wedge dy$ is exact. We write it as $\omega = d\eta$ for the one-form

$$\eta = x \; dy.$$

Note that this is certainly not the only choice of η, but any η such that $d\eta = \omega$ will work.

Then, by Green's theorem, we know that the area A of the polygon is given by

$$\begin{aligned}
A &= \int_D dx \wedge dy \\
&= \int_{\partial D} \eta \\
&= \sum_{i=1}^n \int_{\alpha_i} \eta,
\end{aligned}$$

where in the last line α_i is a parametrization of the line segment L_i. More explicitly, since the line segment L_i joins the points (x_i, y_i) and (x_{i+1}, y_{i+1}), we can parametrize it as $\alpha_i : [0,1] \to \mathbb{R}^2$ with

$$\alpha_i(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i)).$$

Then

$$\alpha_i^* \eta = (x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i)\ dt.$$

The line integral along α_i is then:

$$\begin{aligned}\int_{\alpha_i} \eta =& \int_0^1 (x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i)\ dt \\ =&(y_{i+1} - y_i)\left(x_i + \frac{1}{2}(x_{i+1} - x_i)\right) \\ =&\frac{1}{2}(y_{i+1} - y_i)(x_i + x_{i+1}) \\ =&\frac{1}{2}(x_i y_{i+1} - x_{i+1} y_i + x_{i+1} y_{i+1} - x_i y_i).\end{aligned}$$

Finally, we add up these line integrals to get the area of the polygon. We get:

$$\begin{aligned}A =& \sum_{i=1}^n \int_{\alpha_i} \eta \\ =&\frac{1}{2}\sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i + x_{i+1} y_{i+1} - x_i y_i) \\ =&\frac{1}{2}((x_1 y_2 - x_2 y_1 + x_2 y_2 - x_1 y_1) + (x_2 y_3 - x_3 y_2 + x_3 y_3 - x_2 y_2) + \ldots \\ &+ (x_n y_1 - x_1 y_n + x_1 y_1 - x_n y_n)),\end{aligned}$$

where in the last line we used the fact that $x_{n+1} = x_1$ and $y_{n+1} = y_1$. We see that the terms $x_1 y_1, x_2 y_2, \ldots$ all cancel out, and we are left with

$$A = \frac{1}{2}((x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \ldots + (x_n y_1 - x_1 y_n)).$$

Bingo!

5. We already studied the one-form

$$\eta = -\frac{y}{x^2 + y^2}\ dx + \frac{x}{x^2 + y^2}\ dy,$$

defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$, since it is the typical example of a one-form that is closed but not exact. For instance, we proved in Exercise 3.4.3.2 that it is not exact by showing that its line integral along a circle of radius one around the origin is non-vanishing. In this problem we show that the line integral of η is in fact non-vanishing for every simple closed curve that encloses the origin, and vanishing for every simple closed curve that does not pass through or enclose the origin.

(a) Consider an arbitrary simple closed curve C_0 with canonical orientation that does not pass through or enclose the origin. Use Green's theorem to show that

$$\int_{C_0} \eta = 0.$$

(b) Let C_1 be an arbitrary simple closed curve with canonical orientation that encloses the origin. Explain why the argument of (a) does not apply here. So we need to do something else to study the line integral of ω along C_1.

(c) As in (b), let C_1 be an arbitrary simple closed curve with canonical orientation that encloses the origin. Suppose that K is a circle centered at the origin, with a radius small enough that K lies completely inside C_1. Give K a counterclockwise orientation. Use Green's theorem to show that

$$\int_{C_1} \eta = \int_K \eta.$$

(d) Using part (c), show that it implies that

$$\int_{C_1} \eta = 2\pi.$$

You have showed that the line integral of η along an arbitrary simple closed curve that encloses the origin is non-vanishing (and is in fact equal to 2π), while the line integral along an arbitrary simple closed curve that does not pass through or enclose the origin is vanishing. Isn't it neat?

Solution. The key in this problem is to be very careful with the domain of definition of the one-form η. The one-form is define on the open set $U = \mathbb{R}^2 \setminus \{(0,0)\}$.

As a starting point, recall that the one-form η is closed, that is, $d\omega = 0$. Indeed,

$$\begin{aligned} d\eta =& \left(-\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right) dy \wedge dx + \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \right) dx \wedge dy \\ =& \frac{(x^2-y^2)+(y^2-x^2)}{(x^2+y^2)^2} \, dx \wedge dy \\ =& 0. \end{aligned}$$

This will be useful in this problem.

(a) We assume that C_0 is a simple closed curve that does not pass through or enclose the origin, with canonical orientation. Therefore, $C_0 \subset U$. Moreover, if we denote by D_0 the region consisting of the closed curve C_0 and its interior, then $D_0 \subset U$ as well, since the origin is not in D_0. Therefore, by Green's theorem,

$$\begin{aligned} \int_{C_0} \eta =& \int_{D_0} d\eta \\ =& 0, \end{aligned}$$

since $d\eta = 0$.

(b) If C_1 is a simple closed curve that encloses the origin, we cannot apply the argument in (a). The reason is that, if we denote by D_1 the region consistin of the closed curve C_1 and its interior, then D_1 is not a subset of U, since D_1 contains the origin. In other words, the one-form η is not defined on D_1, which is one of the assumptions in the statement of Green's theorem. Therefore, Green's theorem does not apply. (Indeed, as we will see, if Green's theorem applied, it would imply that the line integral of η along C_1 vanishes, which is not true, as we will see.)

(c) We suppose that C_1 is an arbitrary simple closed curve with canonical orientation that encloses the origin, and that K is a circle centered at the origin that lies completely inside C_1, also oriented counterclockwise. For instance, C_1 could be the orange curve in the figure below, and K the blue circle.

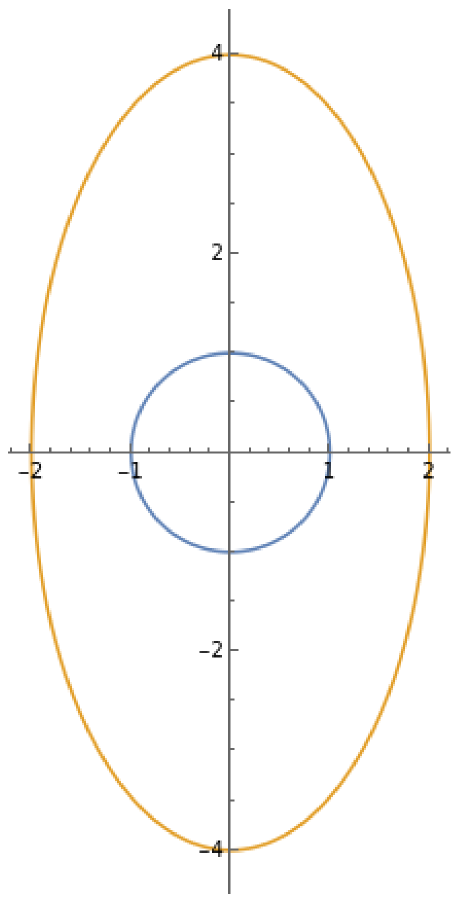


Figure 5.7.9 An example of curves C_1 and K.

We let D_1 be the region bounded by the two closed curves C_1 and K, and we give D_1 a canonical orientation. By definition, the origin is not in D_1, so $D_1 \subset U$ and the one-form is well defined on D_1. We can then apply Green's theorem. The boundary of D_1 has two components, $\partial D_1 = C_1 \cup K$, and if D_1 is oriented counterclockwise, then the induced orientation on the boundary is counterclockwise for the outer boundary C_1 and clockwise for the inner boundary K. To clarify the notation, we write $(C_1)_+$ for C_1 with counterclockwise orientation, K_+ for K with counterclockwise orientation, and

K_- for K with clockwise orientation. So we can write the oriented boundary of D_1 as $\partial D_1 = (C_1)_+ \cup K_-$. By Green's theorem, we have:

$$\begin{aligned}\int_{D_1} d\eta =& \int_{\partial D_1} \eta \\ =& \int_{(C_1)_+} \eta + \int_{K_-} \eta \\ =& \int_{(C_1)_+} \eta - \int_{K_+} \eta,\end{aligned}$$

where we used the fact that line integrals pick a sign if we change the orientation. But the left-hand-side of the above equation is necessarily zero, since $d\eta = 0$. Therefore,

$$\int_{(C_1)_+} \eta = \int_{K_+} \eta,$$

which is the statement that we were trying to prove, with both curves canonically oriented.

(d) The power of what we did in (c) is to replace the evaluation of the line integral of η along an arbitrary simple closed curve C_1 that encloses the origin by the evaluation of the line integral of η along a circle K centered at the origin of a certain non-zero radius. But we can evaluate the latter explicitly! Indeed, if K has radius R, for some $R \in \mathbb{R}_{>0}$, then we can parametrize K by $\alpha : [0, 2\pi] \to \mathbb{R}^2$ with

$$\alpha(\theta) = (R\cos(\theta), R\sin(\theta)).$$

The pullback of η is

$$\begin{aligned}\alpha^*\eta =& \frac{1}{R^2}\left(-R\sin(\theta)(-R\sin(\theta)) + R\cos(\theta)(R\cos(\theta))\right) d\theta \\ =& d\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_\alpha \eta =& \int_{[0,2\pi]} \alpha^*\eta \\ =& \int_0^{2\pi} d\theta \\ =& 2\pi.\end{aligned}$$

So the line integral of η along any circle K centered at the origin is always non-zero and equal to 2π. By (c), we thus conclude that

$$\int_{C_1} \eta = 2\pi$$

for arbitrary simple closed curves C_1 that enclose the origin. Very powerful result!

5.8 Stokes' theorem

Step 6, part 2: we continue our study of integration of exact two-forms. But this time we consider surface integrals, that is integrals of exact two-forms over parametric surfaces. The result is Stokes' theorem.

Objectives

You should be able to:

- State Stokes' theorem for integrals of exact two-forms over parametric surfaces in $\mathbb{R}^3$.
- Rephrase Stokes' theorem in terms of the associated vector fields.
- Show that Stokes' theorem implies that the integral of an exact two-form over a closed surface in $\mathbb{R}^3$ vanishes.
- Use Stokes' theorem to evaluate surface integrals of exact two-forms.
- Use Stokes' theorem to evaluate line integrals of one-forms over closed curves in $\mathbb{R}^3$.
- Use Stokes' theorem and its consequences to show that a given two-form cannot be exact.

5.8.1 Stoke's theorem

In the previous section we studied integrals of exact two-forms over bounded regions in $\mathbb{R}^2$. We now look at surface integrals of exact two-forms over parametric surfaces in $\mathbb{R}^3$.

As in the previous section, we start by recalling the corresponding statement for curves, which we first studied in Theorem 3.4.1, and revisited in Theorem 5.1.8. Without getting into the details again, the key statement is that the integral of an exact one-form $\omega = df$ over a parametric curve α satisfies the property

$$\int_\alpha df = \int_{\partial\alpha} f,$$

which is the Fundamental Theorem of line integrals rewritten using integrals of zero-forms. The integral on the left-hand-side is a line integral along the oriented parametric curve α, while on the right-hand-side we have the integral of the zero-form f over the oriented boundary $\partial\alpha$ of the parametric curve α.

Looking back at Theorem 3.4.1, the proof of the Fundamental Theorem of line integrals basically amounted to pulling back to the interval $[a, b]$ using the definition of line integrals, and then using the Fundamental Theorem of Calculus. We now do the same thing for surface integrals: we pull back to the region D and then use Green's theorem. The result is known as Stokes' theorem.

Theorem 5.8.1 Stokes' theorem. *Let η be a one-form on an open subset $U \subseteq \mathbb{R}^3$, and $\alpha : D \to \mathbb{R}^3$ a parametric surface whose image $S = \alpha(D) \subset U$ is orientable and oriented by the parametrization. Let $\partial\alpha$ be the boundary of the surface S with its induced orientation, as in Definition 5.5.5 (if the image surface is closed, $\partial\alpha$ is the empty set). Then the integral of* *the exact two-form $d\eta$ along the parametric surface α can be evaluated as:*

$$\int_\alpha d\eta = \int_{\partial\alpha} \eta.$$

In other words, the surface integral of the exact two-form $\omega = d\eta$ along the parametric surface α is equal to the line integral of the one-form η along its oriented boundary curve.

Proof. Stokes' theorem pretty much follows from Green's theorem by pullback. More precisely, by definition of the surface integral,

$$\int_\alpha d\eta = \int_D \alpha^*(d\eta).$$

We know that the exterior derivative commutes with pullback, see Lemma 4.7.4: $\alpha^*(d\eta) = d(\alpha^*\eta)$. So we can rewrite the surface integral as:

$$\int_\alpha d\eta = \int_D d(\alpha^*\eta).$$

This is now the integral of the exact two-form $d(\alpha^*\eta)$ over a closed bounded region D. By Green's theorem, Theorem 5.7.1, we know that

$$\int_D d(\alpha^*\eta) = \int_{\partial D} \alpha^*\eta,$$

where on the right-hand-side we have a line integral of the one-form $\alpha^*\eta$ along the boundary of the closed region $D \subset \mathbb{R}^2$.

We are almost done, but not quite: the last step is actually quite subtle. We would like to say that

$$\int_{\partial D} \alpha^*\eta \stackrel{?}{=} \int_{\partial\alpha} \eta.$$

If the parametrization α is injective everywhere, then it maps the boundary ∂D one-to-one to the boundary ∂S, and so this equality follows from the definition of line integrals, thinking of α restricted to the boundary ∂D as a parametric curve (or a union of parametric curves). However, in general, it is a little more subtle; as we remarked in Remark 5.4.3, $\partial S \subseteq \alpha(\partial D)$, but the two are not necessarily equal.[1] The general proof is beyond the scope of these notes. In any case, the conclusion would be that

$$\int_\alpha d\eta = \int_{\partial\alpha} \eta,$$

which is the statement of Stokes' theorem. ∎

Just as for the Fundamental Theorem of line integrals, there are two direct consequences of Stokes' theorem. First, the integrals of an exact two-form over two surfaces S_1 and S_2 that share the same oriented boundary are equal (this statement for surface integrals is analogous

[1] In fact, not only can α send boundary points in ∂D to points not on the boundary of S, but it can also map whole curves in ∂D to points in S, as it does for instance when parametrizing a sphere with spherical coordinates (two of the boundary lines of the rectangular regions are mapped to the north and south poles of the sphere).

to path independence for line integrals of exact one-forms). Second, the integral of an exact two-form along a closed surface is zero (this statement for surface integrals is analogous to the statement that the line integral of an exact one-form along a closed curve vanishes). We state those as the following corollaries.

Corollary 5.8.2 The surface integrals of an exact two-form along two surfaces that share the same oriented boundary are equal. *Let η be a one-form on $U \subseteq \mathbb{R}^3$. If $\alpha : D_1 \to \mathbb{R}^3$ and $\beta : D_2 \to \mathbb{R}^3$ are two parametric surfaces whose image surfaces $S_1 = \alpha(D_1) \subset U$ and $S_2 = \beta(D_2) \subset U$ share the same boundary $\partial S_1 = \partial S_2$, with the same induced orientation, then*

$$\int_\alpha d\eta = \int_\beta d\eta.$$

Corollary 5.8.3 The surface integral of an exact two-form along a closed surface vanishes. *Let η be a one-form on $U \subseteq \mathbb{R}^3$. If $\alpha : D \to \mathbb{R}^3$ is a parametric surface whose image surface $S = \alpha(D) \subset U$ is closed (that is, it has no boundary, which means that it is itself the boundary of a volume), then*

$$\int_\alpha d\eta = 0.$$

Remark 5.8.4 We can read Stokes' theorem in two ways, just like Green's theorem.

1. The integral of an exact two-form $\omega = d\eta$ over the parametric surface $\alpha : D \to \mathbb{R}^3$ is equal to the line integral of the one-form η over the oriented boundary $\partial\alpha$ of the parametric surface.

2. The integral of a one-form η over a closed oriented simple curve ∂S in $\mathbb{R}^3$ is equal to the surface integral of its exterior derivative $d\eta$ over any surface $S \subset \mathbb{R}^3$ whose boundary is ∂S, with the appropriate orientation.

These two different readings highlight the two different types of applications of Stokes' theorem: either to calculate surface integrals via the first reading, or to calculate line integrals via the second reading. We note that Corollary 5.8.2 can also be useful computationally, to replace a complicated surface integral by an easier one.

Example 5.8.5 Using Stokes' theorem to evaluate a surface integral by transforming it into a line integral. Use Stokes' theorem to evaluate the surface integral of the two-form $\omega = 2z\ dy \wedge dz + \ dx \wedge dy$ along the part of the sphere $x^2 + y^2 + z^2 = 25$ that is above the plane $z = 3$, with orientation given by an outwards pointing normal vector.

The surface is shown in the following figure:

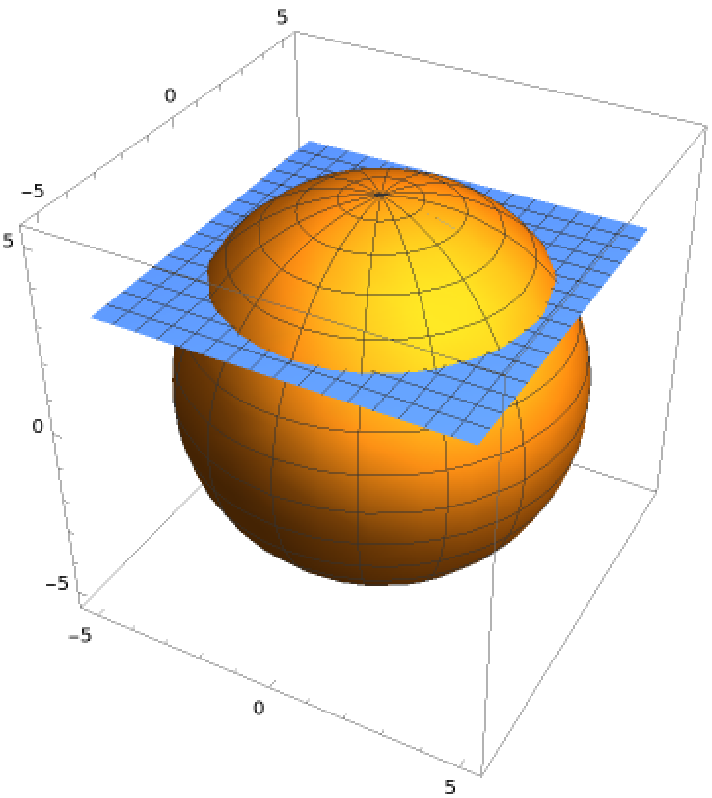


Figure 5.8.6 The surface is part of the sphere of radius 5 above the plane $z = 3$.

To start with, for Stokes' theorem to apply, the two-form $\omega = 2z\ dy \wedge dz + \ dx \wedge dy$ must be exact. But it easy to see that

$$\eta = (x - z^2)\ dy \qquad \Rightarrow \qquad d\eta = \omega,$$

and thus ω is exact.

The boundary ∂S of the surface is the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 3$. Setting $z = 3$ in the equation of the sphere, we get the equation of the circle $x^2 + y^2 = 25 - 9 = 16$. The boundary is thus the circle $x^2 + y^2 = 16$ in the plane $z = 3$. Since we orient the sphere with an outwards pointing normal vector, we see that must walk counterclockwise along the circle, with our heads pointing outside the sphere, to keep the region on our left. Therefore, the oriented boundary is the circle $x^2 + y^2 = 16$ in the plane $z = 3$ with counterclockwise orientation. Stokes' theorem then tells us that

$$\int_S d\eta = \int_{\partial S} \eta.$$

To evaluate the line integral on the right-hand-side we parametrize the oriented boundary with $\alpha : [0, 2\pi] \to \mathbb{R}^3$,

$$\alpha(\theta) = (4\cos(\theta), 4\sin(\theta), 3).$$

We calculate the pullback of the one-form η:

$$\alpha^*\eta = (4\cos(\theta) - 9)(4\cos(\theta))\ d\theta$$

$$=4(4\cos^2(\theta) - 9\cos(\theta))\ d\theta.$$

The line integral is then:

$$\begin{aligned}\int_\alpha \eta =& \int_{[0,2\pi]} \alpha^* \eta \\ =& 4\int_0^{2\pi} (4\cos^2(\theta) - 9\cos(\theta))\ d\theta \\ =& 16\pi,\end{aligned}$$

where we used the fact that $\int_0^{2\pi} \cos(\theta) = 0$ and $\int_0^{2\pi} \cos^2(\theta) = \pi$.

Therefore, by Stokes' theorem, we conclude that

$$\int_S d\eta = 16\pi.$$

□

Example 5.8.7 Using Stokes' theorem to evaluate a surface integral by using a simpler surface. Let us consider the same integral as in the previous example, Example 5.8.5. That is, we want to evaluate the surface integral of the two-form $\omega = 2z\ dy \wedge dz + \ dx \wedge dy$ along the part of the sphere $x^2 + y^2 + z^2 = 25$ that is above the plane $z = 3$, with orientation given by an outwards pointing normal vector.

We can use Stokes' theorem in a different way to evaluate this surface integral. Since we know that the two-form is exact (as $\omega = d\eta$ with $\eta = (x - z^2)\ du$), we know that its surface integral along any two surfaces S and S' that share the same oriented boundary are equal. So instead of integrating over the part of the sphere above the $z = 3$ plane, we could replace this surface by a simpler surface that shares the same boundary, which in this case is the circle $x^2 + y^2 = 16$ in the plane $z = 3$, as we saw in Example 5.8.5. In particular, we could take the surface S' to be the disk $x^2 + y^2 \leq 16$ in the plane $z = 3$, with orientation given by a normal vector pointing upwards. By Stokes' theorem, we know that

$$\int_S \omega = \int_{S'} \omega,$$

so we can evaluate the surface integral over the oriented disk instead.

To do so, we parametrize the disk by $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 4], \theta \in [0, 2\pi]\}$$

and

$$\alpha(r, \theta) = (r\cos(\theta), r\sin(\theta), 3).$$

It is easy to see that the normal vector $\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta$ points upwards, as required. We calculate the pullback two-form:

$$\begin{aligned}\alpha^*\omega =& 2(3)(\sin(\theta)\ dr + r\cos(\theta)\ d\theta) \wedge (0) + (\cos(\theta)\ dr - r\sin(\theta)\ d\theta) \wedge (\sin(\theta)\ dr + r\cos(\theta)\ d\theta) \\ =& r\ dr \wedge d\theta.\end{aligned}$$

The surface integral is then:

$$\int_\alpha \omega = \int_D \alpha^* \omega$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^4 r \ dr d\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=4} d\theta \\
&= 8 \int_0^{2\pi} d\theta \\
&= 16\pi.
\end{aligned}
$$

Fortunately, this is the same answer that we found in Example 5.8.5, as it should! □

Example 5.8.8 Using Stokes' theorem to evaluate a line integral by transforming it into a surface integral. Use Stokes' theorem to evaluate the line integral of the one-form $\omega = xy \ dx + yz \ dy + z^4 \ dz$ over the curve C that forms the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ with $x \geq 0$, $y \geq 0$, $z \geq 0$. Assume that C is given the orientation as moving from the x-axis to the y-axis to the z-axis.

It may be hard to visualize the curve and the region at first. A picture is worth a thousand words. Here is a figure representing the part of the paraboloid $z = 1 - x^2 - y^2$ with $x \geq 0$, $y \geq 0$, $z \geq 0$.

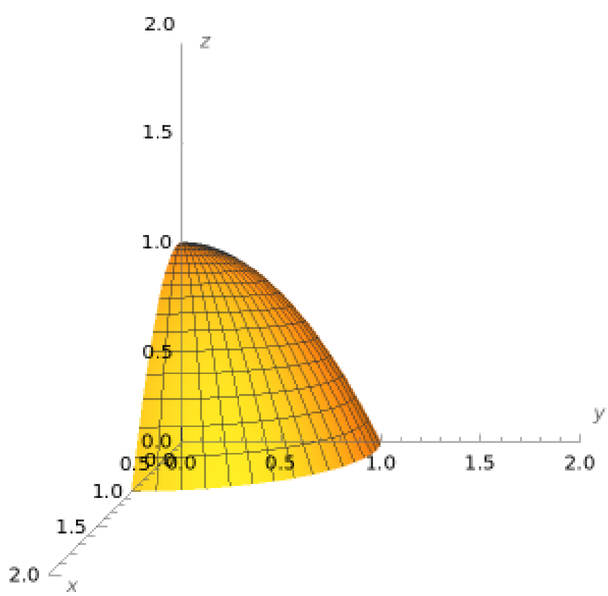


Figure 5.8.9 The part of the paraboloid $z = 1 - x^2 - y^2$ with $x, y, z \geq 0$. The curve is the boundary of the region.

We want to evaluate the line integral of ω along the boundary C of this region. We could evaluate the line integral directly. Or we can use Stokes' theorem, which says that

$$\int_C \omega = \int_S d\omega,$$

where S is the part of the paraboloid described above. We need to be careful with orientation here. We assume that C is oriented going from x to y to z. Looking at the figure, we see that if we walk along the boundary curve in this direction, to keep the surface on the left we must have our heads pointing away from the origin. So the surface S should be oriented with a normal vector pointing in that direction.

To evaluate the surface integral of $d\omega$ along S, we first calculate the exterior derivative. We get:

$$d\omega = x dy \wedge dx + y dz \wedge dy.$$

Next, we parametrize the surface S as $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 1], \theta \in [0, \pi/2]\}$$

and

$$\alpha(r, \theta) = (r \cos(\theta), r \sin(\theta), 1 - r^2).$$

The region D was chosen such that $x(r, \theta), y(r, \theta), z(r, \theta) \geq 0$ on D. Thus it covers the part of the paraboloid that we are interested in.

The normal vector is

$$\begin{aligned}\mathbf{n} =& \mathbf{T}_r \times \mathbf{T}_\theta \\ =& (\cos(\theta), \sin(\theta), -2r) \times (-r \sin(\theta), r \cos(\theta), 0) \\ =& (2r^2 \cos(\theta), 2r^2 \sin(\theta), r).\end{aligned}$$

As its z-component is positive, we see that it points away from the origin, as it should. So the orientation induced by the parametrization is the correct one.

Finally, we calculate the pullback of the two-form $d\omega$:

$$\begin{aligned}\alpha^*(d\omega) =& r \cos(\theta)(\sin(\theta)\ dr + r \cos(\theta)\ d\theta) \wedge (\cos(\theta)\ dr \\ & - r \sin(\theta)\ d\theta) + r \sin(\theta)(-2r\ dr) \wedge (\sin(\theta)\ dr + r \cos(\theta)\ d\theta) \\ =& (-r^2 \cos(\theta) - 2r^3 \sin(\theta) \cos(\theta)) dr \wedge d\theta.\end{aligned}$$

We then evaluate the surface integral:

$$\begin{aligned}\int_\alpha d\omega =& \int_D \alpha^*(d\omega) \\ =& \int_0^{\pi/2} \int_0^1 (-r^2 \cos(\theta) - 2r^3 \sin(\theta) \cos(\theta))\ dr d\theta \\ =& \int_0^{\pi/2} \left[-\frac{r^3}{3} \cos(\theta) - \frac{r^4}{2} \sin(\theta) \cos(\theta) \right]_{r=0}^{r=1} d\theta \\ =& \int_0^{\pi/2} \left(-\frac{1}{3} \cos(\theta) - \frac{1}{2} \sin(\theta) \cos(\theta) \right) d\theta \\ =& -\frac{1}{3} - \frac{1}{4} \\ =& -\frac{7}{12}.\end{aligned}$$

Therefore, by Stokes' theorem, the lintegral of ω along C is

$$\int_C \omega = -\frac{7}{12}.$$

It is a good exercise to evaluate the line integral directly along the boundary curve (which you will need to split into three parametric curves). You should get the same answer $-7/12$. And, as complicated as the calculation above looked, it is actually simpler than calculating the line integral directly! □

Remark 5.8.10 We can also use Stokes' theorem to show that a two-form is not exact, by showing that its surface integral along a closed surface does not vanish. This is similar to what we did for one-forms using the Fundamental Theorem of line integrals, see Exercise 3.4.3.2 (and also Exercise 5.7.3.5). We use Stokes' theorem in this way in Exercise 5.8.3.5 for the two-form in Remark 4.6.6.

5.8.2 Stokes' theorem in terms of vector fields

As usual we translate Stokes' theorem in terms of the associated vector fields, as in Corollary 5.6.5.

Theorem 5.8.11 Stokes' theorem for vector fields. *Let $\mathbf{F}$ be a smooth vector field on $U \subseteq \mathbb{R}^3$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface whose image surface $S = \alpha(D) \subset U$ is orientable and oriented by parametrization, with oriented boundary $\partial\alpha$. Then Stokes' theorem is the statement that*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where on the left-hand-side we used the shorthand notation introduced in Remark 5.6.6 to denote the surface integral of the vector field $\nabla \times \mathbf{F}$ along the oriented parametric surface S, and on the right-hand-side we use the shorthand notation from Remark 3.3.8 to denote the line integral of the vector field $\mathbf{F}$ along the oriented boundary curve ∂S parametrized by $\mathbf{r}$.

From Corollary 5.8.2 and Corollary 5.8.3, we conclude that:

- The surface integral of the curl of a vector field is the same for any two surfaces that share the same oriented boundary.
- The surface integral of the curl of a vector field along a closed surface always vanishes.

As before, we can use Stokes' theorem in both directions. For instance, if we are interested in calculating the line integral of a vector field along a closed curve in $\mathbb{R}^3$ (say, to calculate the work done by a force field on an object moving along the closed curve), then we can use Stokes' theorem, which tells us that that the line integral is equal to the surface integral of the curl of the vector field on any surface whose boundary is the closed curve. In particular, we are free to choose a surface that is simple enough so that the resulting surface integral is easy to evaluate.

5.8.3 Exercises

1. Let S be the part of the cone $z = 2\sqrt{x^2+y^2}$ below the plane $x+z=1$, oriented with an upward normal vector. Let S' be the part of the plane $x+z=1$ contained within the cone $z = 2\sqrt{x^2+y^2}$, also oriented with an upward normal vector. Let η be the one-form

$$\eta = e^{x^2}\ dx + e^{y^3}\ dy + 5(x+z-1)^2\ dz.$$

Use Stokes' theorem to show that

$$\int_S d\eta = 0.$$

Solution. First, the oriented boundary of S is the closed curve at the intersection of the cone $z = 2\sqrt{x^2+y^2}$ and the plane $x+z=1$, oriented counterclockwise (looking from above). We notice that the boundary of S' is exactly the same. Therefore, by Stokes' theorem, we know that

$$\int_S d\eta = \int_{S'} d\eta.$$

So we may as well evaluate the integral of $d\eta$ along the surface S', which lies within the plane $x+z=1$.

To do so, we need to evaluate the exterior derivative $d\eta$. We get:

$$\begin{aligned} d\eta =& d(e^{x^2}) \wedge dx + d(e^{y^3}) \wedge dy + 5d((x+z-1)^2) \wedge dz \\ =& 10(x+z-1)\ dx \wedge dz. \end{aligned}$$

To evaluate the surface integral $\int_{S'} d\eta$, we would need to parametrize the surface S', pullback $d\eta$, and evaluate the integral as a double integral. However, we can show that the integral will be zero right away without going through the whole calculation. Indeed, when we pullback $d\eta$ by the parametrization, we need to evaluate its component function on the surface. But the surface S' lies within the plane $x+z=1$, and we see right away that the component function vanishes when $x+z-1$. In other words, the two-form $d\eta = 0$ when restricted to the plane $x+z=1$. So it will certainly vanish on S'. Therefore, we can conclude right away that

$$\int_{S'} d\eta = 0,$$

and, by Stokes' theorem,

$$\int_S d\eta = 0.$$

2. Consider the force field

$$\mathbf{F}(x,y,z) = (-yx^2 + z, xy^2, e^{xy} + z).$$

Use Stokes' theorem to find the work done by the force field when moving an object counterclockwise along the circle $x^2+y^2=4$ in the plane $z=10$.

Solution. Let ∂S be the circle $x^2+y^2=4$ in the plane $z=10$ with counterclockwise orientation, and let S be the disk $x^2+y^2 \leq 4$ within the plane $z=10$, with orientation given by an upward pointing normal vector. By Stokes' theorem, we know that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}.$$

So instead of evaluating the line integral of $\mathbf{F}$ along ∂S, we can evaluate the surface integral of the curl of $\mathbf{F}$ along S.

First, we calculate the curl of $\mathbf{F}$. We get:

$$\boldsymbol{\nabla} \times \mathbf{F} = (xe^{xy}, 1 - ye^{xy}, y^2 + x^2).$$

To calculate the surface integral, we need to parametrize the surface S. We use the following parametrization: $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 2], \theta \in [0, 2\pi]\},$$

with

$$\alpha(r, \theta) = (r\cos(\theta), r\sin(\theta), 10).$$

The tangent vectors are

$$\mathbf{T}_r = (\cos(\theta), \sin(\theta), 0), \qquad \mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 0).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, r),$$

which points upward as required. Therefore, the surface integral is

$$\begin{aligned}
\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_D (\boldsymbol{\nabla} \times \mathbf{F})(\alpha(r, \theta)) \cdot \mathbf{n}\ dA \\
&= \iint_D (r\cos(\theta)e^{r^2\sin(\theta)\cos(\theta)}, 1 - r\sin(\theta)e^{r^2\sin(\theta)\cos(\theta)}, r^2) \cdot (0, 0, r)\ dA \\
&= \iint_D r^3\ dA \\
&= \int_0^{2\pi} \int_0^2 r^3\ dr d\theta \\
&=4 \int_0^{2\pi} d\theta \\
&=8\pi.
\end{aligned}$$

Therefore, by Stokes' theorem we conclude that the work done by the force field when moving an object counterclockwise along the circle ∂S is 8π.

3. Let S be the surface consisting of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$, with orientation given by an outward pointing normal vector, and $\mathbf{F}$ the vector field

$$\mathbf{F} = (xy, zx, \cos(x^2) + z^5).$$

Use Stokes' theorem to evaluate the surface integral

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}.$$

Solution. We can use Stokes' theorem in two different ways here: either to rewrite the surface integral as a line integral, or to rewrite it as a surface integral over a simpler surface. For completeness I will do it both ways.

First, we can use Stokes' theorem to rewrite the surface integral as a line integral. Let us do this first. We notice that the boundary curve ∂S of the surface S is the square with vertices $(\pm 1, \pm 1, -1)$, since the bottom side of the cube is not included in S. The induced orientation on the boundary amounts to moving counterclockwise along the square, as seen from above. Thus, by Stokes' theorem, we know that

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where the right-hand-side is the line integral of the vector field $\mathbf{F}$ along the square with counterclockwise orientation.

To evaluate the line integral, we split the square into four line segments L_i, $i = 1, \ldots, 4$. We parametrize the line segments by $\alpha_i : [0, 1] \to \mathbb{R}^3$ with

$$\begin{aligned} \alpha_1(t) =&(-1 + 2t, -1, -1), \\ \alpha_2(t) =&(1, -1 + 2t, -1), \\ \alpha_3(t) =&(1 - 2t, 1, -1), \\ \alpha_4(t) =&(-1, 1 - 2t, -1). \end{aligned}$$

We calculate the tangent vectors:

$$\begin{aligned} \mathbf{T}_1 &= (2, 0, 0), \\ \mathbf{T}_2 &= (0, 2, 0), \\ \mathbf{T}_3 &= (-2, 0, 0), \\ \mathbf{T}_4 &= (0, -2, 0). \end{aligned}$$

We finally evaluate the line integrals:

$$\begin{aligned} \int_{L_1} \mathbf{F}(\alpha_1(t)) \cdot \mathbf{T}_1 \ dt =& 2 \int_0^1 (-1 + 2t)(-1) \ dt = 0, \\ \int_{L_2} \mathbf{F}(\alpha_2(t)) \cdot \mathbf{T}_2 \ dt =& 2 \int_0^1 (-1)(1) \ dt = -2, \\ \int_{L_3} \mathbf{F}(\alpha_3(t)) \cdot \mathbf{T}_3 \ dt =& -2 \int_0^1 (1 - 2t)(1) \ dt = 0, \\ \int_{L_4} \mathbf{F}(\alpha_4(t)) \cdot \mathbf{T}_4 \ dt =& -2 \int_0^1 (-1)(-1)/dt = -2. \end{aligned}$$

Adding up these four line integrals, we get -4. We conclude, by Stokes' theorem, that the surface integral

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = -4.$$

Another approach would have been to use Stokes' theorem to rewrite the surface integral as a simpler surface integral. For instance we could have replaced the surface S by the missing side of the cube (namely the bottom), as it shares the same boundary with S. Let us call the bottom side of the cube S'. If we give it an upward pointing normal vector, it induces the same orientation on the boundary as S. Thus, by Stokes' theorem,

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}'.$$

To evaluate the integral on the right-hand-side, we first calculate the curl of $\mathbf{F}$. We get:

$$\boldsymbol{\nabla} \times \mathbf{F} = \Big(-x, 2x \sin(x^2), z - x\Big).$$

Next, we need to parametrize S', which is the square with vertices $(\pm 1, \pm 1, -1)$. We write $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [-1, 1], v \in [-1, 1]\},$$

and

$$\alpha(u, v) = (u, v, -1).$$

The tangent vectors are

$$\mathbf{T}_u = (1, 0, 0), \qquad \mathbf{T}_v = (0, 1, 0),$$

with normal vector

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = (0, 0, 1).$$

This is the correct orientation. We thus have

$$\begin{aligned}
\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}' &= \int_{-1}^{1} \int_{-1}^{1} (-u, 2u \sin(u^2), -1 - u) \cdot (0, 0, 1) \ du dv \\
&= -\int_{-1}^{1} \int_{-1}^{1} (u + 1) \ du dv \\
&= -4.
\end{aligned}$$

The same answer as before!

4. Let $C \subset \mathbb{R}^2$ be a simple closed curve, and $D \subset \mathbb{R}^2$ the region consisting of C and its interior. Let $\alpha : D \to \mathbb{R}^3$ be the parametric surface

$$\alpha(u, v) = (u, v, 1 - u - v),$$

with $S = \alpha(D)$ and ∂S the boundary curve of S. Let η be the one-form

$$\eta = -z \ dx - 2x \ dy + 4y \ dz.$$

Use Stokes' theorem to show that the line integral of η along ∂S is equal to the area of the region $D \subset \mathbb{R}^2$.

Solution. First, by Stokes' theorem we know that

$$\int_{\partial\alpha} \eta = \int_{\alpha} d\eta,$$

with the orientation induced by the parametrization. To evaluate the integral on the right-hand-side, we calculate the exterior derivative:

$$d\eta = -dz \wedge dx - 2dx \wedge dy + 4dy \wedge dz.$$

We calculate its pullback:

$$\begin{aligned}\alpha^*(d\eta) =& -(-du - dv) \wedge du - 2du \wedge dv + 4dv \wedge (-du - dv)\\ =& du \wedge dv.\end{aligned}$$

Therefore, the surface integral becomes

$$\begin{aligned}\int_\alpha d\eta =& \int_D \alpha^*(d\eta)\\ =& \int_D du \wedge dv,\end{aligned}$$

which is simply the area of the region D. We then conclude, by Stokes' theorem, that the line integral of η along ∂S is equal to the area of the region D.

5. Consider the two-form

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy),$$

from Remark 4.6.6. It is defined on $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$.

(a) Show that ω is closed, that is, $d\omega = 0$.

(b) Show that ω is not exact by showing that the surface integral of ω along the sphere $x^2+y^2+z^2 = 1$ is non-zero.

(c) Does that contradict Poincare's theorem for two-forms?

Solution. (a) To show that it is closed, we calculate the exterior derivative. We get:

$$\begin{aligned}d\omega =& \Big(\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &+ \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\\ &+ \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\Big) dx \wedge dy \wedge dz\\ =& \frac{1}{(x^2+y^2+z^2)^{5/2}}\Big(3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)\Big)\ dx \wedge dy \wedge dz\\ =& 0.\end{aligned}$$

Therefore, ω is closed.

(b) In fact, we already did such a surface integral in Exercise 5.6.4.5, but for completeness we redo it here. We parametrize the sphere as always with spherical coordinates, $\alpha : D \to \mathbb{R}^3$,

$$D = \{(\theta, \phi) \in \mathbb{R}^2 \mid \theta \in [0, \pi], \phi \in [0, 2\pi]\},$$

and

$$\alpha(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)).$$

The pullback of ω is:

$$\begin{aligned}
\alpha^*\omega =& \frac{1}{1}\Big(\sin(\theta)\cos(\phi)(\cos(\theta)\sin(\phi)d\theta + \sin(\theta)\cos(\phi)d\phi) \wedge (-\sin(\theta)d\theta) \\
&+ \sin(\theta)\sin(\phi)(-\sin(\theta)d\theta)) \wedge (\cos(\theta)\cos(\phi)d\theta) - \sin(\theta)\sin(\phi)d\phi) \\
&+ \cos(\theta)(\cos(\theta)\cos(\phi)d\theta) - \sin(\theta)\sin(\phi)d\phi) \wedge (\cos(\theta)\sin(\phi)d\theta + \sin(\theta)\cos(\phi)d\phi)\Big) \\
=& \Big(\sin^3(\theta)\cos^2(\phi) + \sin^3(\theta)\sin^2(\phi) + \sin(\theta)\cos^2(\theta)\Big)\, d\theta \wedge d\phi \\
=& \sin(\theta)d\theta \wedge d\phi.
\end{aligned}$$

We finally calculate the surface integral:

$$\begin{aligned}
\int_\alpha \omega =& \int_D \alpha^*\omega \\
=& \int_0^{2\pi}\int_0^\pi \sin(\theta)d\theta d\phi \\
=& 2\int_0^{2\pi} d\phi \\
=& 4\pi.
\end{aligned}$$

The result is certainly non-zero!

As the surface integral of ω along the sphere is non-zero, we conclude that ω cannot be exact, since we know from Stokes' theorem that the surface of integral of an exact two-form along a closed surface must vanish.

(c) We have shown that ω is a closed two-form that is not exact. But it does not contradict the statement of Poincare's lemma for two-forms, see Theorem 4.6.4, since ω is not defined on all of $\mathbb{R}^3$; it is not defined at the origin. It also does not contradict version II of Poincare's lemma that we saw in Theorem 4.6.5, since ω is also not defined on an open ball (the origin is missing).

6. Let $\mathbf{F}$ be a smooth vector field on $\mathbb{R}^3$ and f a smooth function on $\mathbb{R}^3$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface. Show that

$$\iint_\alpha (f\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = -\iint_\alpha (\boldsymbol{\nabla} f \times \mathbf{F}) \cdot d\mathbf{S} + \int_{\partial\alpha} f\mathbf{F} \cdot d\mathbf{r}.$$

Solution. The key here is recall some of the vector calculus identities that we encountered previously. From Lemma 4.4.9, we know that

$$\boldsymbol{\nabla} \times (f\mathbf{F}) = (\boldsymbol{\nabla} f) \times \mathbf{F} + f\boldsymbol{\nabla} \times \mathbf{F}.$$

From the point of view of differential forms, this identity follows from the graded product rule for the exterior derivative.

Using this identity, we can write:

$$\iint_\alpha (f\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = -\iint_\alpha (\boldsymbol{\nabla} f) \times \mathbf{F} + \iint_\alpha \boldsymbol{\nabla} \times (f\mathbf{F}).$$

The second term on the right-hand-side is the surface integral of the curl of the vector field $f\mathbf{F}$. Using Stokes' theorem, we can rewrite it as a line integral:

$$\iint_{\alpha} \boldsymbol{\nabla} \times (f\mathbf{F}) = \int_{\partial\alpha} f\mathbf{F} \cdot d\mathbf{r}.$$

Therefore, we conclude that

$$\iint_{\alpha} (f\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = -\iint_{\alpha} (\boldsymbol{\nabla} f) \times \mathbf{F} + \int_{\partial\alpha} f\mathbf{F} \cdot d\mathbf{r}.$$

5.9 Applications of surface integrals

In this section we study some applications of surface integrals, such as calculating the flux of a vector field across a surface.

Objectives

You should be able to:

- Determine and evaluate appropriate surface integrals and flux integrals in the context of applications in science.

5.9.1 Surface integrals as flux integrals

The main interpretation of surface integrals consists in calculating the flux of a vector field across a surface in the direction of the normal vector. The easiest way to understand what this means is in terms of fluid mechanics.

Suppose that the vector field $\mathbf{v}(x, y, z)$ is the velocity field of a fluid. Suppose that the function $\rho(x, y, z)$ is the mass density of the fluid. Thus the vector field $\rho\mathbf{v}$ is the rate of flow (mass per unit time) per unit area. We would like to calculate the rate of flow (mass per unit time) of fluid crossing a surface S in the normal direction $\mathbf{n}$. How can we do that?

We use the famous "divide and conquer", or "slice it till you make it" process of integral calculus. We divide the surface S into tiny pieces of surface, and calculate the rate of flow through these tiny pieces of surface. Then we "sum over tiny pieces of surface", and take the limit of an infinite number of pieces with infinitesimal size, which turns the calculation into a double integral.

More precisely, let dS be the area of a tiny piece of surface at (x, y, z) on S. Assuming that S is a parametric surface, let

$$\hat{\mathbf{n}} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|}$$

be the normalized normal vector at this point. Thus $\rho\mathbf{v} \cdot \hat{\mathbf{n}}$ is the rate of flow per unit area in the normal direction at the point (x, y, z). Therefore, the rate of flow of fluid through this tiny piece of surface in the normal direction is

$$\rho\mathbf{v} \cdot \hat{\mathbf{n}} dS.$$

But... what is the area dS? We will come back to this in Section 7.2. Suppose that the tiny region of D that is mapped to the tiny piece of surface by the parametrization is a rectangle, with sides of lengths du and dv. We can think of these two sides as being vectors in the u and v directions with length du and dv. We can think of these vectors as being mapped by the parametrization to the rescaled tangent vectors $du\mathbf{T}_u$ and $dv\mathbf{T}_v$. These two vectors span a parallelogram in the tangent plane, with area given by $|du\mathbf{T}_u \times dv\mathbf{T}_v| = |\mathbf{T}_u \times \mathbf{T}_v| du dv$. The idea is that the area of this parallelogram is a good approximation of the area dS of the tiny piece of surface, since the tangent plane is a good approximation of the surface. (And, when we sum over tiny pieces of surface and take the limit of the Riemann sum, this approximation will become exact.) As a result, we can write

$$dS = |\mathbf{T}_u \times \mathbf{T}_v| dA$$

for the area of the tiny piece of surface, with $dA = du dv$. Therefore, we get that the rate of flow of fluid through this tiny piece of surface in the normal direction is:

$$\begin{aligned} \rho \mathbf{v} \cdot \hat{\mathbf{n}} |\mathbf{T}_u \times \mathbf{T}_v| dA =& \rho \mathbf{v} \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|} |\mathbf{T}_u \times \mathbf{T}_v| dA \\ =& \rho \mathbf{v} \cdot \mathbf{n} dA, \end{aligned}$$

where $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ (not normalized).

The final step is to sum over tiny pieces of surfaces and take the limit of an infinite number of pieces of surface of infinitesimal area, which turns the sum into a double integral. The result is the double integral

$$\iint_D \rho \mathbf{v} \cdot \mathbf{n} \, dA,$$

which we recognize as the surface integral of the vector field $\rho\mathbf{v}$! Therefore, the surface of integral of $\rho\mathbf{v}$ calculates the rate of flow of fluid across the surface in the normal direction.

While this was formulated for fluid velocity, one can study similar processes for other vector fields. The result is called the "flux" of the vector field.

Definition 5.9.1 The flux of a vector field across a surface. Let $\mathbf{F}(x, y, z)$ be a vector field on $U \subseteq \mathbb{R}^3$, and $\alpha : D \to \mathbb{R}^3$ an oriented parametric surface with normal vector $\mathbf{n}$. The surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\alpha(u, v)) \cdot \mathbf{n} \, dA$$

is called the **flux of $\mathbf{F}$ across S in the normal direction $\mathbf{n}$**. ♢

This is of course just the standard surface integrals that we have studied already. But it gives it an interpretation as calculating the flux of the vector field, which is why surface integrals are also known as "flux integrals".

5.9.2 Flux integrals beyond fluids

The main motivation for introducing the notion of flux is to calculate the rate of flow of a fluid across a surface. But the concept of flux is also important in other physical applications.

One such example is in electromagnetism. Suppose that $\mathbf{E}(x, y, z)$ is an electric field. Then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

calculates what is known as the **electric flux** across the surface S. In fact, one of the important laws in electromagnetism is **Gauss's law**, which relates the electric charge to the flux of an electric field. More precisely, if S is a closed surface in an electric field $\mathbf{E}$, Gauss's law states that the net charge Q enclosed by the surface S is given by

$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S},$$

i.e. it is the electric flux through S rescaled by a constant ϵ_0 known as the "permittivity of free space". Here the surface S should be given the orientation of an outward normal vector.

Example 5.9.2 The electric flux and net charge of a point source. Let q be a point charge at the origin. The electric force follows an inverse square law, that is, the magnitude of the electric field produced by the charge is inversely proportional to the square of the distance from the charge. More precisely, the electric field produced by the point charge is the vector field

$$\mathbf{E}(x, y, z) = \frac{q}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{3/2}}(x, y, z).$$

Now suppose that you want to calculate the electric flux produced by the point charge across a sphere of radius R centered at the origin. By Gauss's law, this should calculate the total charge enclosed by the sphere. Since we only have a point charge at the origin, with charge q, we expect the electric flux to be given by q. Is that what we get?

The electric flux across the sphere in the outward direction is given by the flux integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}.$$

We already calculated such surface integrals in Exercise 5.6.4.5 using spherical coordinates. The result of this exercise applies here, with $C = \frac{q}{4\pi\epsilon_0}$. We thus conclude that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi \frac{q}{4\pi\epsilon_0} = \frac{q}{\epsilon_0}.$$

In particular, Gauss's law states that the total charge enclosed by the sphere is

$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \frac{q}{\epsilon_0} = q.$$

Phew!

We note that the flux does not depend on the radius of the sphere, as it should. In fact, we can go further. An argument very similar to Exercise 5.7.3.5 holds here as well, but using the divergence theorem (which we will explore in Section 6.2) instead of Green's theorem. The conclusion of the argument is that the flux of the electric field of a point charge at the origin across any closed surface that encloses the origin, not just the sphere, is always equal to q, as it should by Gauss's law. Cool! □

The concept of flux is also used for instance in the study of heat flow. Suppose that the temperature at a point (x, y, z) in a substance is given by the function $T(x, y, z)$. Then the **heat flow** is given the gradient of the temperature function, rescaled by a constant. More precisely, the heat flow is

$$\mathbf{F} = -K\boldsymbol{\nabla} T,$$

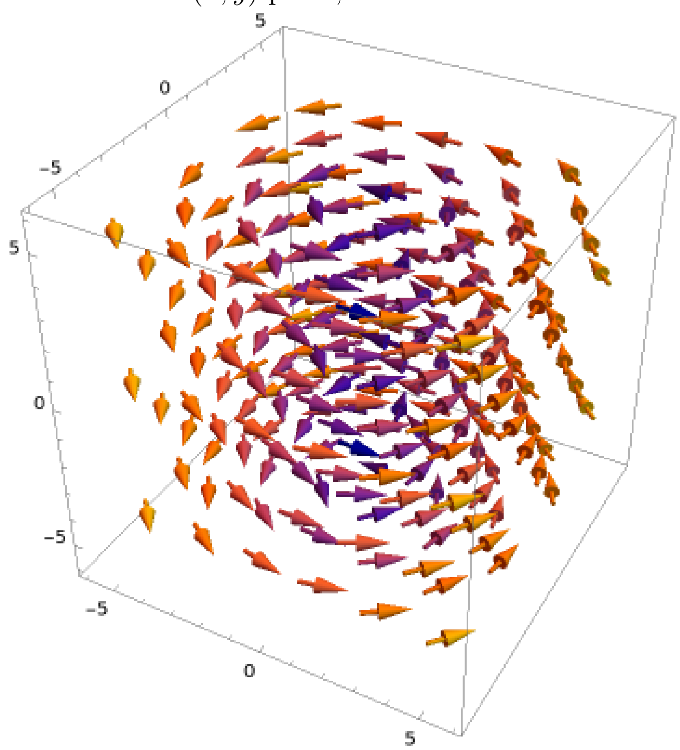
where K is a constant called the conductivity of the substance. We are then often interested in calculating the rate of heat flow across a surface S, which is given by the flux of the vector field $\mathbf{F}$ across S:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \boldsymbol{\nabla} T \cdot d\mathbf{S}.$$

5.9.3 Exercises

1. Consider a fluid moving with velocity $\mathbf{v}(x, y, z) = (-y, x, 0)$ and constant mass density $\rho(x, y, z) = \rho_0$ with ρ_0 a positive constant. (As we saw in Exercise 2.1.3.2, this type of fluid motion is a vortex in the (x, y)-plane.) Show that the rate of flow of fluid across the cylinder $x^2 + y^2 = R^2$, with $-a \leq z \leq a$, for some positive constants a, R, is zero. Explain why this result makes sense, looking at the fluid motion.

Solution. As we saw in Exercise 2.1.3.2, this type of fluid motion is a vortex in the (x, y)-plane. Below is, first, a sketch of the velocity vector field of the fluid in 3 dimensions. But since there is no motion in the z-direction, I also present a two-dimensional plot of the vector field in the (x, y)-plane, which makes the vortex motion more



explicit.

Figure 5.9.3 The vector field $\mathbf{v}(x, y, z) = (-y, x, 0)$.

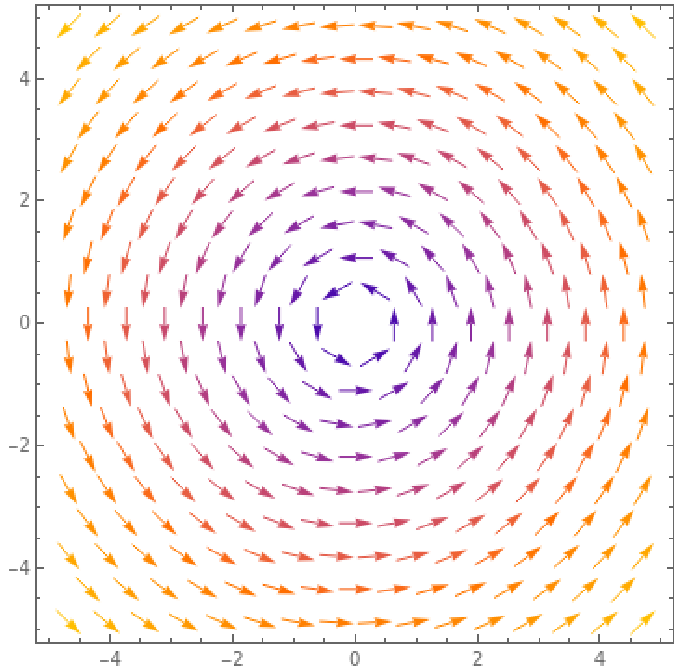


Figure 5.9.4 The vector field $\mathbf{v}(x,y) = (-y, x)$, which corresponds to the projection of the previous vector field in the (x,y)-plane.

Looking at the two-dimensional figure, we see that the motion is moving around in a circular motion in the (x,y)-plane about the origin. In three dimensions, since there is no motion in the z-direction, the fluid flow is circular about the z-axis. As such, if we imagine a cylinder centered on the z-axis in the fluid, we see that there is no fluid flowing across the surface of the cylinder. Thus we expect that the rate of flow of the fluid across the cylinder should be zero. Let us show this.

To calculate the rate of flow across the cylinder S, we need to evaluate the surface integral

$$\iint_S \rho \mathbf{v} \cdot d\mathbf{S}.$$

We parametrize the surface of the cylinder as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u,\theta) \in \mathbb{R}^2 \mid u \in [-a,a], \theta \in [0, 2\pi]\}$$

and

$$\alpha(u,\theta) = (R\cos(\theta), R\sin(\theta), u).$$

The tangent vectors are

$$\mathbf{T}_u = (0,0,1), \qquad \mathbf{T}_\theta = (-R\sin(\theta), R\cos(\theta), 0).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_\theta = (-R\cos(\theta), -R\sin(\theta), 0).$$

It is pointing inward, but it does not matter anyway which orientation we choose as we will show that the integral is zero. The surface integral is

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\alpha(u,\theta))\cdot\mathbf{n}\ dA\\ &= \iint_D (-R\sin(\theta), R\cos(\theta), 0)\cdot(-R\cos(\theta), -R\sin(\theta), 0)\ dA\\ &= \iint_D (R^2\sin(\theta)\cos(\theta) - R^2\sin(\theta)\cos(\theta))\ dA\\ &= 0.\end{aligned}$$

This is just as we expected: there is no rate of flow of the fluid across the cylinder.

2. Find the flux of the vector field

$$\mathbf{F}(x,y,z) = (x,0,z)$$

exiting the solid cone

$$V = \{(x,y,z)\in\mathbb{R}^3 \mid 0\leq z\leq \sqrt{9-x^2-y^2}\}.$$

Solution. To find the flux we need to evaluate the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S},$$

where S is the surface boundary of the solid cone V.

The cone is shown in the figure below:

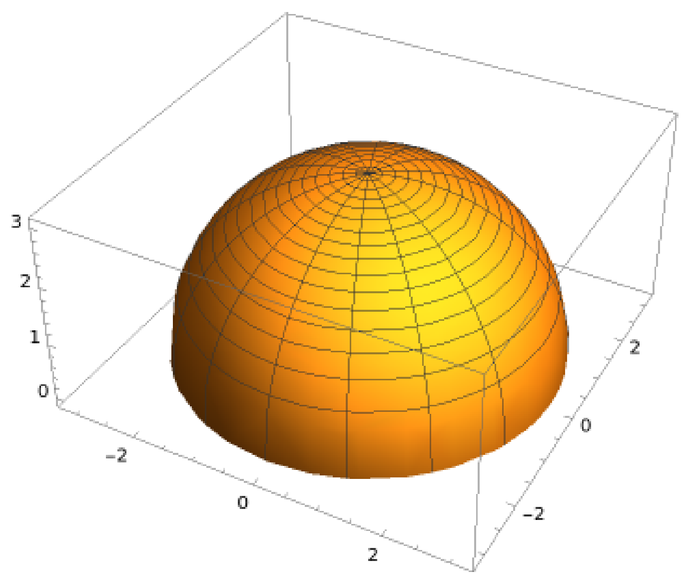


Figure 5.9.5 The solid cone V.

The boundary surface of V has two components: the lateral surface of the cone, which we will call S_1, and the bottom disk, which we will call S_2. We need to evaluate the surface integral on both components, and add up the results to calculate the flux of the vector field exiting the cone.

We parametrize S_1 as $\alpha_1 : D_1 \to \mathbb{R}^3$ with

$$D_1 = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,3], \theta \in [0,2\pi]\},$$

with

$$\alpha(r,\theta) = (r\cos(\theta), r\sin(\theta), \sqrt{9-r^2}).$$

The tangent vectors are

$$\mathbf{T}_r = \left(\cos(\theta), \sin(\theta), -\frac{r}{\sqrt{9-r^2}}\right), \qquad \mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 0).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = \left(\frac{r^2}{\sqrt{9-r^2}}\cos(\theta), \frac{r^2}{\sqrt{9-r^2}}\sin(\theta), r\right).$$

It points upward in the z-direction, and thus outward of the cone, as we want to calculate the flux exiting the cone. The surface integral is then

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}_1 =& \iint_{D_1} \left(r\cos(\theta), 0, \sqrt{9-r^2}\right)\cdot\left(\frac{r^2}{\sqrt{9-r^2}}\cos(\theta), \frac{r^2}{\sqrt{9-r^2}}\sin(\theta), r\right)\, dA\\
=& \iint_{D_1}\left(\frac{r^3}{\sqrt{9-r^2}}\cos^2(\theta) + r\sqrt{9-r^2}\right)\, dA\\
=& \int_0^3\int_0^{2\pi}\left(\frac{r^3}{\sqrt{9-r^2}}\cos^2(\theta) + r\sqrt{9-r^2}\right) d\theta dr\\
=& \pi\int_0^3\left(\frac{r^3}{\sqrt{9-r^2}} + 2r\sqrt{9-r^2}\right)\, dr\\
=& \pi\int_9^0\left(\frac{1}{2}\frac{u-9}{\sqrt{u}} - \sqrt{u}\right)\, du\\
=& \pi\int_9^0\left(-\frac{1}{2}u^{1/2} - \frac{9}{2}u^{-1/2}\right)\, du\\
=& \pi\left[-\frac{1}{3}u^{3/2} - 9u^{1/2}\right]_{u=9}^{u=0}\\
=& 36\pi.
\end{aligned}$$

As for the bottom disk S_2, it is the disk $x^2+y^2 \leq 9$ in the $z=0$ plane. We can parametrize it as $\alpha_2 : D_2 \to \mathbb{R}^3$ with

$$D_2 = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,3], \theta \in [0,2\pi]\},$$

and

$$\alpha(r,\theta) = (r\sin(\theta), r\cos(\theta), 0).$$

The tangent vectors are

$$\mathbf{T}_r = (\sin(\theta), \cos(\theta), 0), \qquad \mathbf{T}_\theta = (r\cos(\theta), -r\sin(\theta), 0).$$

The normal vector is

$$\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, -r),$$

which points downward, i.e. outward, which is what we want. The surface integral is

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}_2 =& \iint_{D_2} (r\sin(\theta), 0, 0) \cdot (0, 0, -r)\ dA \\ =& 0.\end{aligned}$$

Thus there is no flux through the bottom of the cone.

We conclude that the flux exiting the cone is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 36\pi.$$

3. Use Gauss's law to find the net charge enclosed by the closed surface consisting of the cylinder $x^2 + y^2 = 1$, $-2 \leq z \leq 2$, with its top and bottom, in the electric field $\mathbf{E}(x, y, z) = (x, 0, 0)$.

Solution. To calculate the charge, by Gauss's law we need to calculate the eletric flux, that is, the surface integral of the electric field along the lateral surface of the cylinder as well as its top and bottom. To do so, we need to split the surface into three components. However, we can directly conclude that the surface integrals along the top and bottom of the cylinder will be zero. Why? The surface integrals calculate the flux of the vector field in the normal direction. That is, in the integrand we take the dot product of the vector field and the normal vector. For the top and bottom of the cylinder, we know that the normal vector will point in the z-direction (as the cylinder is centered around the z-axis). But the z-component of $\mathbf{E}$ is zero, and therefore the integrand will vanish.

As a result, we only have to consider the lateral surface S of the cylinder, which we parametrize as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, \theta) \in \mathbb{R}^2 \mid u \in [-2, 2], \theta \in [0, 2\pi]\}$$

and

$$\alpha(u, \theta) = (\cos(\theta), \sin(\theta), u).$$

The tangent vectors are

$$\mathbf{T}_u = (0, 0, 1), \qquad \mathbf{T}_\theta = (-\sin(\theta), \cos(\theta), 0),$$

and the normal vector is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_\theta = (-\cos(\theta), -\sin(\theta), 0)$$

This is pointing inward, so we change the sign of the normal vector. The surface integral is then:

$$\begin{aligned}\iint_\alpha \mathbf{E} \cdot d\mathbf{S} =& \iint_D (\cos(\theta), 0, 0) \cdot (\cos(\theta), \sin(\theta), 0)\ dA \\ =& \int_{-2}^{2} \int_0^{2\pi} \cos^2(\theta)\ d\theta du\end{aligned}$$

$$
\begin{aligned}
&= \pi \int_{-2}^{2} du \\
&= 4\pi.
\end{aligned}
$$

We conclude, by Gauss's law, that the net charge enclosed by the surface is

$$Q = \epsilon_0 \iint_\alpha \mathbf{E} \cdot d\mathbf{S} = 4\pi\epsilon_0.$$

4. Suppose that the temperature distribution in a substance in $\mathbb{R}^3$ is given by the function

$$T(x, y, z) = x^2 + xy.$$

Show that the rate of heat flow along any horizontal plane $z = C$, where C is a constant, is zero. Explain why this is consistent with your expectation.

Solution. The heat flow is given by

$$\mathbf{F} = -K\boldsymbol{\nabla}T = -K(2x + y, x, 0).$$

To calculate the rate of heat flow across an horizontal plane $z = C$, we need to calculate the surface integral of the heat flow $\mathbf{F}$ along the plane. In this process, we take the dot product of $\mathbf{F}$ with the normal vector $\mathbf{n}$ to extract the normal component of $\mathbf{F}$. Since the plane is horizontal, its normal vector will point in the z-direction. But the vector field $\mathbf{F}$ has a vanishing z-component; therefore, $\mathbf{F} \cdot \mathbf{n} = 0$, and the surface integral along the horizontal plane will vanish.

This is consistent with our expectation because the temperature distribution does not depend on z. So it does not vary as we move in the z-direction; indeed, its gradient has a vanishing z-component. As a result, there is no heat flowing through horizontal planes, as we found.

Chapter 6

Beyond one- and two-forms

6.1 Generalized Stokes' theorem

So far we have seen a number of theorems that take a similar form: the fundamental theorem of calculus, the fundametal theorem of line integrals, Green's theorem, and Stokes' theorem. In this section we show that they are all special cases of the mother of all integral theorems, the "generalized Stokes' theorem".

Objectives

You should be able to:

- State the generalized Stokes' Theorem.
- Show that for integration of an exact one-form over a closed interval in $\mathbb{R}$, it reduces to the Fundamental Theorem of Calculus.
- Show that for integration of an exact one-form over a parametric curve in $\mathbb{R}^n$, it reduces to the Fundamental Theorem of line integrals.
- Show that for integration of an exact two-form over a closed bounded region in $\mathbb{R}^2$, it reduces to Green's theorem.
- Show that for integration of an exact two-form over a parametric surface in $\mathbb{R}^3$, it reduces to Stokes' theorem.

6.1.1 The generalized Stokes' theorem

So we far we have seen four integral theorems, all related to integration of exact one- and two-forms: the fundamental theorem of calculus, the fundamental theorem of line integrals, Green's theorem, and Stokes' theorem. While at first the theorems look different, you may have noticed that they all take a similar form. In fact, we could write all four integral theorems in the following form:

$$\int_M d\omega = \int_{\partial M} \omega.$$

All that changes is the meaning of ω and M. More precisely:

1. If ω is a zero-form (a function) on $U \subseteq \mathbb{R}$, and $M = [a, b] \subset U$ is an oriented interval, then it becomes the fundamental theorem of calculus, see Theorem 5.1.7.

2. If ω is a zero-form (a function) on $U \subseteq \mathbb{R}^n$, and M is a parametric curve $\alpha : [a, b] \to \mathbb{R}^n$ whose image is in U, then it becomes the fundamental theorem of line integrals, see Theorem 5.1.8.

3. If ω is a one-form on $U \subseteq \mathbb{R}^2$, and $M \subset U$ is a closed bounded oriented region, then it becomes Green's theorem, see Theorem 5.7.1.

4. If ω is a one-form on $U \subseteq \mathbb{R}^3$, and M is a parametric surface $\alpha : D \to \mathbb{R}^3$ whose image is in U, then it becomes Stokes' theorem, see Theorem 5.8.1.

In mathematics, when we see something like this, we dig deeper and try to determine whether it is a coincidence or not that all these integral theorems pretty much take the same form. More often than not, such a coincidence is a hint that there is something going on behind the scenes, that there is a unifying principle at play. This is precisely the case here.

The unifying principle is the mother of all integral theorems, known as the "generalized Stokes' theorem". It states that the relationship above is very general. The precise statement is the following.

Theorem 6.1.1 The generalized Stokes' theorem. *Let M be a k-dimensional oriented manifold and ∂M its boundary (which is a $(k-1)$-dimensional manifold) with the induced orientation. Let ω be a $(k-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

The scope of this theorem is really awe-inspiring, at least in the eye of a mathematician. We will not prove this theorem as it is beyond the scope of the class, but we can try to make sense of it.

We know what a k-form is, at least over open subsets in $\mathbb{R}^n$. The key object that we have not defined and that appears in the statement of the theorem is the notion of a "manifold", which is fundamental in differential geometry. So let us say a few words about manifolds.

6.1.2 An informal introduction to manifolds

The concept of manifold is essential in mathematics and physics to do calculations on complicated geometric spaces. Informally, an n-dimensional manifold M is a space that "locally looks like $\mathbb{R}^n$". What does it mean? It means that for any point $p \in M$, one can find an invertible map that sends an open subset of M around p to an open subset of $\mathbb{R}^n$. This map is called a "coordinate chart"; by mapping the open subset of M to an open subset of $\mathbb{R}^n$, we are basically defining coordinates on the complicated space M. This is the essence of a manifold. The description of most manifolds however requires more than one coordinate charts; we can always map an open subset of M to an open subset of $\mathbb{R}^n$, but we cannot generally map the whole space M to an open subset of $\mathbb{R}^n$, because the global structure of the space M may be quite complicated. The different coordinate charts are then "glued" together in a consistent way, which gives rise to so-called "transition functions", which are basically changes of coordinates between different charts. To be able to do calculus on manifolds, we usually require that these transition functions, or coordinate changes, are differentiable (or even smooth).

In the end, the key feature of a manifold is that locally, instead of doing calculations on the space M itself, you can use the coordinate chart to do calculations on $\mathbb{R}^n$ instead. How do we do this? We use the pullback! Indeed, our coordinate chart is an invertible map, so we can pullback objects on M via the inverse of the coordinate chart to turn them into objects on $\mathbb{R}^n$, where we can do calculus. For instance, using pullback with respect to a coordinate chart, we can define integration of an n-form on a region of an n-dimensional manifold via integration of an n-form over a region in $\mathbb{R}^n$, which is something that we studied in this class. We use once again the fundamental principle of reducing something complicated to something that we already know how to solve, and the pullback is there to help! How neat is this.

Most of the spaces that we encountered in this class, such as parametric curves, parametric surfaces, etc. are examples of manifolds. But the definition of manifolds is much more general. A key feature of manifolds is that they are defined “intrinsically”. When we talked about parametric curves, or parametric surfaces, we introduced complicated geometry, but the way we did it was by embedding a curve or a surface in a higher-dimensional space $\mathbb{R}^m$. For manifolds, you do not need to embed them into higher-dimensional spaces to get interesting geometry; the geometry is intrinsic in the definition of a manifold.

But in the end, you already know many manifolds. Here are a few examples.

- The circle is a one-dimensional manifold, with no boundary.
- The sphere (i.e. the surface of a ball) is a two-dimensional manifold, with no boundary.
- Parametric curves (the way we defined them) are one-dimensional manifolds, possibly with boundary.
- Parametric surfaces are two-dimensional manifolds, possibly with boundary.
- The graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is an n-dimensional manifold.
- Spacetime, where we live, is a manifold!

6.1.3 Back to the generalized Stokes’ theorem

We now understand that an n-dimensional manifold M is basically a complicated looking space that locally looks like $\mathbb{R}^n$. If the space is orientable (as we saw for surfaces in $\mathbb{R}^3$, this is not always obvious), we can choose an orientation on M, like we did for parametric curves and surfaces. The boundary ∂M of M is also a manifold, but one dimension less: it is a $(n-1)$-dimensional manifold. The chosen orientation on M induces an orientation on the boundary ∂M, just like we did again for parametric curves and surfaces.

Even though we haven’t defined integration of forms over manifolds, as we mentioned above it can be done via pullback with respect to coordinate charts, and the result is that integration of forms over manifolds is not much different from what we already did in this class. So we can, at least informally, understand the truly beautiful statement of the generalized Stokes’ theorem:

$$\int_M d\omega = \int_{\partial M} \omega.$$

To end this section, we summarize in a table how the four integral theorems that we already saw arise as special cases of the generalized Stokes' theorem. We add a fifth integral theorem to the table: the divergence theorem, which is the topic of the next section.

Table 6.1.2 Integral theorems as special cases of the generalized Stokes' theorem

M	ω	Integral theorem
Closed interval in $\mathbb{R}$	0-form	Fundamental theorem of calculus
Parametric curve in $\mathbb{R}^n$	0-form	Fundamental theorem of line integrals
Closed bounded region in $\mathbb{R}^2$	1-form	Green's theorem
Parametric surface in $\mathbb{R}^3$	1-form	Stokes' theorem
Closed bounded region in $\mathbb{R}^3$	2-form	Divergence theorem

Next time someone tells you something about the fundamental theorem of calculus, you can reply: "oh, I know this theorem, it's just a special case of the generalized Stokes' theorem"! :-)

6.2 Divergence theorem in $\mathbb{R}^3$

We show that the generalized Stokes' theorem for a closed bounded region in $\mathbb{R}^3$ reduces to the divergence theorem of vector calculus.

Objectives

You should be able to:

- Define integration of three-forms over closed bounded regions in $\mathbb{R}^3$.
- For exact three-forms, rephrase the generalized Stokes' theorem as the divergence theorem in $\mathbb{R}^3$.
- Summarize all the integral theorems of vector calculus as particular cases of the generalized Stokes' theorem.

6.2.1 Integrating a three-form over a region in $\mathbb{R}^3$

Our goal in this section is to study Stokes' theorem for two-/three-forms. As a first step we need to define integration of three-forms over regions in $\mathbb{R}^3$, just like we did for two-forms over regions in $\mathbb{R}^2$ in Section 5.2 and Section 5.3.

In this section we concentrate on solid regions $E \subset \mathbb{R}^3$ that consist of a closed surface and its interior, such as the regiong bounded by a sphere, or a rectangular box. We saw in Definition 5.2.1 how to define the orientation of $\mathbb{R}^n$, and in particular in Example 5.2.5 for $\mathbb{R}^3$, in which case it is given by a choice of right-handed or left-handed twirl, with right-handed twirl being the canonical orientation. We define the orientation of a region in $\mathbb{R}^3$ as being induced from the orientation of the ambient space.

Definition 6.2.1 Orientation of a region in $\mathbb{R}^3$ and induced orientation on the boundary. Let $E \subset \mathbb{R}^3$ be a solid region that consists of a closed surface and its interior. Its boundary ∂E is the closed surface. Choose an orientation on $\mathbb{R}^3$. We define **the orientation**

of E as being the orientation induced by the ambient vector space $\mathbb{R}^3$. We write E_+ for the region with the canonical (right-handed twirl) orientation, and E_- for the opposite (left-handed twirl) orientation.

If E is canonically oriented, we define the **induced orientation on the boundary** ∂E as corresponding to an outward pointing normal vector. ◊

Example 6.2.2 Solid region bounded by a sphere in $\mathbb{R}^3$. Let $E \subset \mathbb{R}^3$ be the solid region in $\mathbb{R}^3$ bounded by the sphere $x^2 + y^2 + z^2 = R^2$. Its boundary ∂E is the sphere itself, that is, the surface $x^2 + y^2 + z^2 = R^2$. If we give E the canonical orientation given by a choice of ordered basis on $\mathbb{R}^3$ corresponding to a right-handed twirl, then the induced orientation on the sphere is that of a normal vector pointing outward. □

With this definition of orientation, we can define the integral of a three-form over a region in $\mathbb{R}^3$.

Definition 6.2.3 Integral of a three-form over a closed bounded region in $\mathbb{R}^3$. Let $E \subset \mathbb{R}^3$ be a solid region that consists of a closed surface and its interior. Let $\omega = f\, dx \wedge dy \wedge dz$ be a three-form on an open subset $U \subseteq \mathbb{R}^3$ that contains E. If E has canonical orientation, we define the **integral of ω over** E as:

$$\int_E \omega = \iiint_E f\ dV,$$

where on the right-hand-side we mean the standard triple integral from calculus. If E has opposite orientation, we define

$$\int_E \omega = -\iiint_E f\ dV.$$

Note that, as for the integral of two-forms over regions, the choice of basic three-form $dx \wedge dy \wedge dz$ (which is consistent with the ordering of the canonical orientation) when expressing ω in terms of the function f is important here, as integrals of three-forms are oriented, while triple integrals are not. (See Remark 5.3.5.) ◊

The definition reduces the evaluation of integrals of three-forms to triple integrals, which you have encountered already in your previous calculus course. We will focus here on regions that are recursively supported, as we did for regions in $\mathbb{R}^2$ (more general regions could be expressed as unions of recursively supported regions). We say that:

1. A region $E \subset \mathbb{R}^3$ is xy**-supported** (also called **type 1**) if there exists a region D in the (x, y)-plane and two continuous functions $z_1(x, y), z_2(x, y)$ such that

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, z_1(x, y) \leq z \leq z_2(x, y)\}.$$

2. A region $E \subset \mathbb{R}^3$ is yz**-supported** (also called **type 2**) if there exists a region D in the (y, z)-plane and two continuous functions $x_1(y, z), x_2(y, z)$ such that

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid (y, z) \in D, x_1(x, z) \leq x \leq x_2(y, z)\}.$$

3. A region $E \subset \mathbb{R}^3$ is xz**-supported** (also called **type 3**) if there exists a region D in the (x, z)-plane and two continuous functions $y_1(x, z), y_2(x, z)$ such that

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid (x, z) \in D, y_1(x, z) \leq y \leq y_2(x, z)\}.$$

If the two-dimensional region $D \subset \mathbb{R}^2$ is also supported on at least one of the two coordinates, we say that the solid region E is **recursively supported**. In this case the triple integral can be evaluated as an interated integral.

We note that in the case of rectangular regions, Fubini's theorem still applies, and the order of integration for the iterated integrals does not matter.

Example 6.2.4 Integral of a three-form over a recursively supported region. Let $E \subset \mathbb{R}^3$ be the solid region bounded by the planes $x = 0$, $x = 2$, $y = 2$, $z = 0$ and $z = y$. Find the integral of the three-form

$$\omega = xyz\ dx \wedge dy \wedge dz$$

over E with canonical orientation.

By definition, we know that

$$\int_E \omega = \iiint_E xyz\ dV.$$

We want to evaluate the triple integral on the right-hand-side.

We can express the region E as an xy-supported region. Indeed, if $D \subset \mathbb{R}^3$ is the rectangular region $D = [0, 2] \times [0, 2]$ in the (x, y)-plane, then

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, 0 \leq z \leq y\}.$$

In fact, using the definition of the rectangular region D, we could write

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x \in [0, 2], y \in [0, 2], 0 \leq z \leq y\},$$

which shows that E is recursively supported. We can then write the triple integral as an iterated integral, and evaluate:

$$\begin{aligned}
\iiint_E xyz\ dV =& \int_0^2 \int_0^2 \int_0^y xyz\ dzdydx \\
=& \int_0^2 \int_0^2 xy \left[\frac{z^2}{2}\right]_{z=0}^{z=y} dydx \\
=& \frac{1}{2} \int_0^2 \int_0^2 xy^3\ dydx \\
=& \frac{1}{2} \int_0^2 x \left[\frac{y^4}{4}\right]_{y=0}^{y=2} dx \\
=& 2 \int_0^2 x\ dx \\
=& 4.
\end{aligned}$$

□

To conclude this section, we should show that our integration theory is invariant under orientation-preserving reparametrizations, just as we did for two-forms in Subsection 5.3.2. We will be brief here and simply state the result. Let $E_1, E2 \subset \mathbb{R}^3$ be recursively supported regions, and let $\phi : E_2 \to E_1$ be a bijective and invertible function (that can be extended to a

C^1 function on an open subset $U \subseteq \mathbb{R}^3$ containing E_2).

Lemma 6.2.5 Integrals of three-forms over regions in $\mathbb{R}^3$ are oriented and reparametrization-invariant. *Let ω be a three-form on an open subset $U \subseteq \mathbb{R}^3$ that contains E_1, and $\phi : E_2 \to E_1$ as above. Then:*

- *If* $\det J_\phi > 0$ *in the interior of* E_2,

$$\int_{E_2} \phi^* \omega = \int_{E_1} \omega.$$

- *If* $\det J_\phi < 0$ *in the interior of* E_2,

$$\int_{E_2} \phi^* \omega = - \int_{E_1} \omega.$$

We will not write the proof here, as it is almost identical to the proof of the corresponding statement for two-forms in Lemma 5.3.7. The key is that the pullback brings forth the determinant of the Jacobian of the transformation, and invariance reduces to the transformation (or change of variables) formula for triple integrals. As for two-forms, one can say that **the transformation formula for triple integrals is simply the statement the integrals of three-forms over regions in $\mathbb{R}^3$ are invariant under orientation-preserving reparametrizations.**

More precisely, just as for two-forms, the transformation formula for triple integrals involves the absolute value of the determinant of the Jacobian, while invariance under pullback for integrals of three-forms involves the determinant of the Jacobian directly. This is because integrals of three-forms are oriented, while triple integrals are not.

6.2.2 The divergence theorem in $\mathbb{R}^3$

Now that we understand integration of three-forms over regions in $\mathbb{R}^3$, we can go back to the generalized Stokes' theorem Theorem 6.1.1 and see what it becomes when M is taken to be a solid region $E \subset \mathbb{R}^3$ that consists of a closed surface and its interior.

Theorem 6.2.6 The divergence theorem in $\mathbb{R}^3$. *Let ω be a two-form on $U \subseteq \mathbb{R}^3$. Let $E \subset U$ be a solid region that consists of a closed surface and its interior, and ∂E its surface boundary. Give E the canonical orientation, and ∂E the induced orientation corresponding to an outward pointing normal vector. Then*

$$\int_E d\omega = \int_{\partial E} \omega,$$

where the integral on the right-hand-side is a surface integral of ω over the boundary ∂E realized as a parametric surface.

In vector calculus language, if $\mathbf{F}$ *is the vector field associated to the two-form ω, then*

$$\iiint_E (\boldsymbol{\nabla} \cdot \mathbf{F})\ dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S},$$

where the integral on the right-hand-side is the surface integral of the vector field $\mathbf{F}$ *over the boundary surface ∂E with normal vector pointing outward.*

We will not prove the divergence theorem here. We note that it follows directly from the generalized Stokes' theorem Theorem 6.1.1, just like our four other integral theorems. The vector calculus translation follows directly from our dictionary between differential forms and vector calculus concepts.

Remark 6.2.7 Just as for Green's theorem, we can read the divergence theorem in two different ways, starting from the left-hand-side or the right-hand-side. This results in two potential applications: either to evaluate the volume integral of the divergence of a vector field (or an exact three-form), or to evaluate the surface integral of a vector field (a two-form) over a closed surface. However, as was the case for Green's theorem, the divergence theorem is mostly useful to evaluate surface integrals over closed surfaces by transforming them into volume integrals over the interior of the region.

Example 6.2.8 Using the divergence theorem to evaluate the flux of a vector field over a closed surface in $\mathbb{R}^3$. Find the flux of the vector field

$$\mathbf{F}(x,y,z) = (xy, \sin(z^2) + y + \cos(x^3), e^{xy})$$

in the outward direction over the surface of the solid region E that lies above the (x,y)-plane and below the surface $z = 2 - x - y^3$, $x \in [-1,1]$, $y \in [-1,1]$.

We could try to evaluate the surface integral directly, but given how complicated the vector field is, this would probably be a nightmare. Or we can use the divergence theorem, which tells us that

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\boldsymbol{\nabla} \cdot \mathbf{F}) dV.$$

The divergence of the vector field is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = y + 1,$$

which is of course much simpler, so using the divergence theorem looks like a good strategy.

To evaluate the volume integral we need to write the solid region E as a recursively supported region. Let us first look at what the solid region looks like. The surface $Z = 2 - x - y^3$ is shown below.

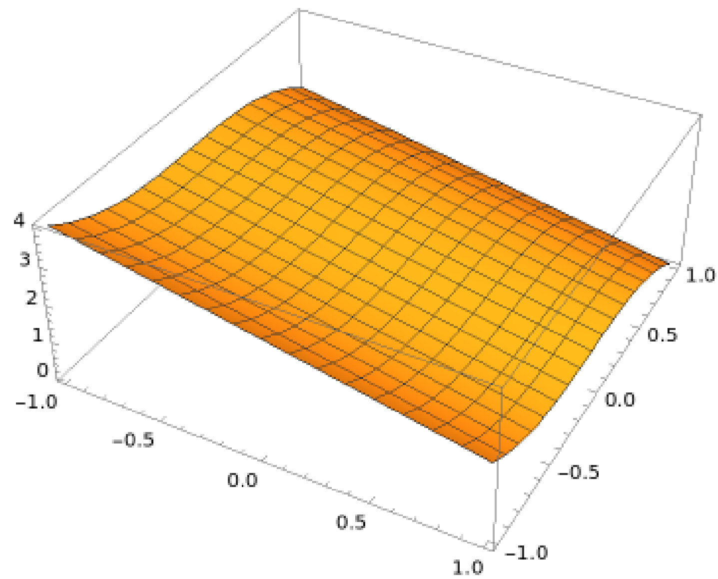


Figure 6.2.9 The surface $z = 2 - x - y^3$ over the rectangle $[-1, 1] \times [-1, 1]$ in the (x, y)-plane.

The solid region E consists of the region bounded by the surface shown above, the four sides of the box in the figure, and the bottom of the box. It can be written as an xy-supported region:

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x \in [-1, 1], y \in [-1, 1], 0 \leq z \leq 2 - x - y^3\}.$$

The volume integral can then be evaluated:

$$\begin{aligned}
\iiint_E (\nabla \cdot \mathbf{F}) dV &= \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{2-x-y^3} (y+1)\ dzdydx \\
&= \int_{-1}^{1} \int_{-1}^{1} (y+1)\,[z]_{z=0}^{z=2-x-y^3}\ dydx \\
&= \int_{-1}^{1} \int_{-1}^{1} (2y - xy - y^4 + 2 - x - y^3)\ dydx \\
&= \int_{-1}^{1} \left[y^2 - \frac{xy^2}{2} - \frac{y^5}{5} + 2y - xy - \frac{y^4}{4} \right]_{y=-1}^{y=1} dx \\
&= \int_{-1}^{1} \left(\frac{18}{5} - 2x \right)\ dx \\
&= \frac{36}{5}.
\end{aligned}$$

Therefore, by the divergence theorem, the flux of $\mathbf{F}$ over the surface ∂E is equal to $36/5$. □

6.2.3 Exercises

1. Use the divergence theorem to find the surface integral of the two-form

$$\omega = 3xy^2\ dy \wedge dz + (e^x z + yz^2)\ dz \wedge dx + xy\ dx \wedge dy$$

over the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$,

$x = -2$, with orientation given by an outward pointing normal vector.

Solution. Let E be the solid region described in the problem, and ∂E its boundary surface with outward pointing normal vector. The divergence theorem tells us that

$$\int_{\partial E} \omega = \int_E d\omega,$$

where the integral on the right-hand-side is over the solid region E with canonical orientation. Thus instead of evaluating the surface integral, we can evaluate the volume integral of the three-form $d\omega$ over the solid region E.

We calculate the exterior derivative:

$$d\omega = (3y^2 + z^2)\ dx \wedge dy \wedge dz.$$

To evaluate the integral of $d\omega$ over E, we describe the solid region E as a recursively supported region. In fact, it is easiest to work in cylindrical coordinates (to be clear, you could do the whole calculation in Cartesian coordinates as well, and you would get the same answer, but I find it easier to work with cylindrical coordinates). So we do the change of coordinates $\phi : E' \to E$ with

$$\phi(u, r, \theta) = (u, r\cos(\theta), r\sin(\theta)).$$

The pullback of $d\omega$ is easily calculated to be:

$$\begin{aligned}\phi^*(d\omega) =&(3r^2\cos^2(\theta) + r^2\sin^2(\theta))r\ du \wedge dr \wedge d\theta\\ =&r^3(1 + 2\cos^2(\theta))\ du \wedge dr \wedge d\theta.\end{aligned}$$

We note that the determinant of the Jacobian is r, which is positive, and hence the integral is invariant under the change of coordinates (pullback). Thus we can rewrite the integral as

$$\begin{aligned}\int_E d\omega &= \int_{E'} \phi^*(d\omega)\\ &= \int_{E'} r^3(1 + 2\cos^2(\theta))\ du \wedge dr \wedge d\theta.\end{aligned}$$

The region E' is easily described as a rectangular region in (u, r, θ):

$$E' = \{(u, r, \theta) \in \mathbb{R}^3 \mid u \in [-2, -1], r \in [0, 1], \theta \in [0, 2\pi]\}.$$

The volume integral can finally be evaluated:

$$\begin{aligned}\int_{E'} r^3(1 + 2\cos^2(\theta))\ du \wedge dr \wedge d\theta =& \int_{-2}^{-1}\int_0^1\int_0^{2\pi} r^3(1 + 2\cos^2(\theta)) d\theta dr du\\ =&4\pi \int_{-2}^{-1}\int_0^1 r^3\ dr du\\ =&\pi \int_{-2}^{-1} du\\ =&\pi.\end{aligned}$$

Therefore, the integral of ω over the surface ∂E specified in the question is equal to π.

2. Use the divergence theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = (z + \sin(y), y, e^x)$$

across (in the outward direction) the sphere $x^2 + y^2 + z^2 = 16$.

Solution. Let E be the sphere and its interior, and ∂E be the sphere with outward pointing normal vector. The divergence theorem tells us that

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \boldsymbol{\nabla} \cdot \mathbf{F} \, dV,$$

where the integral on the right-hand-side is with canonical orientation.

We first calculate the divergence of $\mathbf{F}$:

$$\boldsymbol{\nabla} \cdot \mathbf{F} = 0 + 1 + 0 = 1.$$

Thus the integral that we are interested in is

$$\iiint_E \boldsymbol{\nabla} \cdot \mathbf{F} \, dV = \iiint_E \, dV,$$

which is simply the volume of the solid region E. As E is the interior of the sphere of radius 4, we know right away that its volume is

$$\frac{4}{3}\pi(4^3) = \frac{256\pi}{3},$$

so we could conclude right away that the flux of the vector field across the sphere ∂E is equal to $\frac{256\pi}{3}$.

3. Use the divergence theorem to evaluate the surface integral of the two-form

$$\omega = (3x^3 + yz) \, dy \wedge dz + (y^3 + xz) \, dz \wedge dx + (3zy^2 + x^7) \, dx \wedge dy$$

over the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the (x, y)-plane, with orientation given by an inward pointing normal vector.

Solution. First, we note that the problem is asking to evaluate the surface integral with a normal vector pointing inward. Thus, if we denote the solid region by E and its boundary surface by ∂E the divergence theorem tells us that

$$\int_{\partial E_-} \omega = -\int_E d\omega,$$

where on the left-hand-side we mean the surface integral with normal vector pointing inward, while on the right-hand-side we mean the volume integral with canonical orientation.

We calculate the exterior derivative:

$$d\omega = (9x^2 + 3y^2 + 3y^2) \, dx \wedge dy \wedge dz = 3(3x^2 + 2y^2) \, dx \wedge dy \wedge dz.$$

We can either work in Cartesian or cylindrical coordinates here. But cylindrical coordinates will make the calculation much easier (believe me: I first did it in Cartesian coordinates, and it is no fun! See below. :-). So we do the change of coordinates $\phi : E' \to E$ with

$$\phi(r, \theta, u) = (r\cos(\theta), r\sin(\theta), u).$$

The pullback of $d\omega$ is easily calculated to be:

$$\begin{aligned}\phi^*(d\omega) =& 3(3r^2\cos^2(\theta) + 2r^2\sin^2(\theta))r\ dr \wedge d\theta \wedge du.\\ =& 3r^3(2 + \cos^2(\theta))\ dr \wedge d\theta \wedge du.\end{aligned}$$

We note that the determinant of the Jacobian is r, which is positive, and hence the integral is invariant under the change of coordinates (pullback). Thus we can rewrite the integral as

$$\begin{aligned}\int_E d\omega =& \int_{E'} \phi^*(d\omega)\\ =& \int_{E'} 3r^3(2 + \cos^2(\theta))\ dr \wedge d\theta \wedge du.\end{aligned}$$

The region E' is easily described as a recursively supported region in (r, θ, u):

$$E' = \{(r, \theta, u) \in \mathbb{R}^3 \mid r \in [0, 1], \theta \in [0, 2\pi], 0 \leq u \leq 1 - r^2\}.$$

The volume integral becomes

$$\begin{aligned}\int_{E'} \phi^*(d\omega) =& 3\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r^3(2 + \cos^2(\theta))\ du d\theta dr\\ =& 3\int_0^1 \int_0^{2\pi} r^3(2 + \cos^2(\theta))(1 - r^2)\ d\theta dr\\ =& 15\pi \int_0^1 (r^3 - r^5)\ dr\\ =& 15\pi \left(\frac{1}{4} - \frac{1}{6}\right)\\ =& \frac{5\pi}{4}.\end{aligned}$$

Therefore, by the divergence theorem the surface integral of the two-form that we are asked to evaluate is equal to minus this result (the surface integral is with respect to a normal vector pointing inward, as mentioned above), and thus is equal to $-\frac{5\pi}{4}$.

For completeness, let me do the calculation in Cartesian coordinates as well -- you will see how much uglier it is. The solid is given by the region enclosed by the parabola $z = 1 - x^2 - y^2$ above the disk $x^2 + y^2 = 1$ in the (x, y)-plane. We describe the solid region as a recursively supported region:

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x \in [-1, 1], -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, 0 \leq z \leq 1 - x^2 - y^2\}.$$

The volume integral becomes:

$$\int_E d\omega = 3\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (3x^2 + 2y^2)\ dz dy dx$$

$$
\begin{aligned}
=&3\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3x^2+2y^2)(1-x^2-y^2)\ dydx\\
=&3\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3x^2-3x^4+(-5x^2+2)y^2-2y^4)\ dydx\\
=&3\int_{-1}^{1}\left[y(3x^2-3x^4+\frac{1}{3}(-5x^2+2)y^2-\frac{2}{5}y^4)\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\ dx\\
=&6\int_{-1}^{1}\sqrt{1-x^2}\left(3x^2-3x^4+\frac{1}{3}(-5x^2+2)(1-x^2)-\frac{2}{5}(1-x^2)^2\right)\ dx\\
=&6\int_{-1}^{1}\sqrt{1-x^2}\left(-\frac{26}{15}x^4+\frac{22}{15}x^2+\frac{4}{15}\right)\ dx\\
=&6\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\left(-\frac{26}{15}\sin^4(\theta)+\frac{22}{15}\sin^2(\theta)+\frac{4}{15}\right)d\theta\\
=&\frac{5\pi}{4}.
\end{aligned}
$$

Here I used the trigonometric substitution $x=\sin(\theta)$, and to evaluate the last trigonometric integral one needs to use a whole bunch of trigonometric identities (or a computer algebra system :-). We fortunately get the same result as before for the volume integral, and we conclude as before that the surface integral is equal to minus this result, that is, $-\frac{5\pi}{4}$.

4. Show that the volume V of a solid region $E\subset\mathbb{R}^3$ bounded by a closed surface ∂E can be written as

$$V=\int_{\partial E}\ x\ dy\wedge dz,$$

where the surface integral is evaluated with the orientation given by an outward pointing normal vector.

Solution. We know that the volume of the region E is given by the integral of the basic three-form $\omega=dx\wedge dy\wedge dz$ over E with canonical orientation:

$$V=\int_E dx\wedge dy\wedge dz.$$

But $\omega=dx\wedge dy\wedge dz$ is exact, as it can be written as $\omega=d\eta$ for the two-form $\eta=x\ dy\wedge dz$. Therefore, by the divergence the theorem, we know that

$$\int_E dx\wedge dy\wedge dz=\int_{\partial E}x\ dy\wedge dz,$$

where the integral on the right-hand-side is the surface integral with orientation given by an upward pointing normal vector.

5. We studied the two-form

$$\omega=\frac{1}{(x^2+y^2+z^2)^{3/2}}(x\ dy\wedge dz+y\ dz\wedge dx+z\ dx\wedge dy)$$

in Remark 4.6.6 and Exercise 5.8.3.5. ω is defined on $U=\mathbb{R}^3\setminus\{(0,0,0)\}$. We proved in Exercise 5.8.3.5 that ω is closed, but that it is not exact, by showing that its surface integral along the sphere $x^2 + y^2 + z^2 = 1$ is equal to 4π. In this problem we show that the surface integral of ω is non-vanishing for all closed surface that contain the origin (and always equal to 4π), while it vanishes for all closed surfaces that do not pass through or enclose the origin.

(a) Consider an arbitrary closed surface S_0 with outward pointing normal vector that does not contain or pass through the origin. Use the divergence theorem to show that
$$\int_{S_0} \omega = 0.$$

(b) Let S_1 be an arbitrary closed surface with outward pointing normal vector that contains the origin. Explain why the argument of (a) does not apply here.

(c) Let S_1 be an arbitrary closed surface with outward pointing normal vector that contains the origin. Let K be a sphere centered at the origin, with a radius small enough that it is contained completely inside S_1. Give K the orientation of a normal vector pointing outward (outward of the sphere K). Use the divergence theorem to show that
$$\int_{S_1} \omega = \int_K \omega.$$

(d) Using part (c), show that it implies that
$$\int_{S_1} \omega = 4\pi.$$

You have shown that the surface integral of ω along any closed surface that contains the origin is 4π, while the surface integral of ω along any closed surface that does not enclose or pass through the origin is zero. Note that this is the argument that is needed to show that, using Gauss's law, the total charge contained within any closed surface that encloses a point charge q at the origin is always equal to q -- see Example 5.9.2.

Solution. (a) We know that the two-form ω is closed, that is, $d\omega = 0$. Furthermore, ω is defined on $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$. If the surface S_0 does not contain or pass through the origin, then the surface S_0 and the solid region E_0 bounded by S_0 lie within U, the domain of definition of ω. Therefore, by the divergence theorem,
$$\int_{S_0} \omega = \int_{E_0} d\omega = 0.$$

(b) If S_1 contains the origin in its interior, we cannot apply the divergence theorem as in (a), since the solid region E_1 bounded by S_1 does not lie within U, the domain of definition of ω.

(c) We stated the divergence theorem only for solid regions that consisted of a single closed surface and its interior, but it in fact applies to any closed bounded region. One simply needs to make sure that each bounding surface is oriented with an outward pointing normal vector, where "outward" means away from the solid region bounded by the surfaces.

In particular, we can consider the solid region E that is inside S_1, but outside the sphere K. It is bounded by the two closed surfaces S_1 and K. As the origin is inside K, it is not within the solid region E. Therefore, $E \subset U$, and the divergence theorem applies to this solid region. In this case, the divergence theorem says that the sum of the surface integral over the outer boundary S_1 with outward pointing normal vector and the inner boundary K with vector pointing away from the solid region (which means that it is pointing inside the sphere K) is equal to the volume integral over E, which is zero since $d\omega = 0$:

$$\int_{S_1,\text{out}} \omega + \int_{K,\text{in}} \omega = \int_E d\omega = 0.$$

We conclude that

$$\int_{S_1,\text{out}} \omega = -\int_{K,\text{in}} \omega.$$

To get rid of the minus sign, we can reverse the orientation on K, and consider a normal vector that points outward of the sphere K. We get:

$$\int_{S_1,\text{out}} \omega = \int_{K,\text{out}} \omega,$$

as stated in the question.

(d) We already calculated the surface integral of ω along the sphere $x^2 + y^2 + z^2 = 1$ in Exercise 5.8.3.5 and obtained the result 4π. In fact, the same calculation for a sphere of arbitrary radius $x^2 + y^2 + z^2 = R^2$ would still give 4π (in fact we already did this calculation in Exercise 5.6.4.5). Therefore, by (c), we conclude that the surface integral of ω along an arbitrary closed surface S_1 that contains the origin, with outward pointing normal vector, is:

$$\int_{S_1} \omega = 4\pi.$$

6.3 Divergence theorem in $\mathbb{R}^n$

We show that the divergence theorem holds in $\mathbb{R}^n$, not just in $\mathbb{R}^3$. It follows again from the generalized Stokes' theorem, but we need to rewrite it a little bit to see this.

Objectives

You should be able to:

- Use the Hodge star operator to rewrite the generalized Stokes' theorem in $\mathbb{R}^n$, which can be rewritten as the divergence theorem in $\mathbb{R}^n$.
- Formulate and use the divergence theorem in $\mathbb{R}^n$ to calculate integrals.

6.3.1 A divergence theorem in $\mathbb{R}^n$?

In the previous section, we showed that the generalized Stokes' theorem, in the particular case where ω is a two-form on $\mathbb{R}^3$ and M is a solid region $E \subset \mathbb{R}^3$, reduces to the divergence theorem in $\mathbb{R}^3$, which reads

$$\iiint_E (\nabla \cdot \mathbf{F})\ dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}.$$

Contrary to Green's and Stokes' theorem, the divergence theorem involves the divergence of the vector field, not the curl. While the notion of curl of a vector field is not so easy to generalize to $\mathbb{R}^n$, the divergence can be generalized easily.

More precisely, let $\mathbf{F}(x_1, \ldots, x_n) = (f_1, \ldots, f_n)$ be a smooth vector field on $U \subseteq \mathbb{R}^n$, with $f_1, \ldots, f_n : U \to \mathbb{R}$ smooth functions. We can define the divergence of $\mathbf{F}$ naturally as

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \ldots + \frac{\partial f_n}{\partial x_n}.$$

Now suppose that $E \subset \mathbb{R}^n$ is a closed bounded region that consists of a closed $(n-1)$-dimensional space ∂E and its interior. The integral of $\nabla \cdot \mathbf{F}$ over E is defined naturally in calculus as a multiple ("n-tuple") integral, which can be rewritten as an iterated integral if E is recursively supported. The "surface" integral over ∂E can also be generalized; since ∂E is a $(n-1)$-dimensional subspace in $\mathbb{R}^n$, it has a well defined normal vector. If E is canonically oriented (choose the canonical ordered basis on $\mathbb{R}^n$), we say that the induced orientation on ∂E corresponds to an outward pointing normal vector, as for $\mathbb{R}^3$. A natural question then arise: does the divergence theorem generalize to any dimension? That is, is it true that

$$\underbrace{\int \cdots \int_E}_{n \text{ times}} (\nabla \cdot \mathbf{F})\ dV_n = \underbrace{\int \cdots \int_{\partial E}}_{(n-1) \text{ times}} (\mathbf{F} \cdot \mathbf{n}) dV_{n-1}$$

where the integral on the left-hand-side is an n-tuple integral over the region $E \subset \mathbb{R}^n$, and the right-hand-side is an integral of the vector field $\mathbf{F}$ over the parametrized surface ∂E with normal vector pointing outward?

The answer is yes, and it again follows from the generalized Stokes' theorem. But we need to rewrite the generalized Stokes' theorem a little bit to see this.

6.3.2 Rewriting the generalized Stokes' theorem

Let us recall the generalized Stokes' theorem from Theorem 6.1.1:

$$\int_M d\omega = \int_{\partial M} \omega,$$

where M is an oriented n-dimensional manifold, ∂M its boundary, and ω a $(n-1)$-form.

Let us focus on a case similar to the previous section, where we take ∂E to be a closed $(n-1)$-dimensional space in $\mathbb{R}^n$ (such as a closed surface in $\mathbb{R}^3$ in the previous section), and E to be the n-dimensional region of $\mathbb{R}^n$ consisting of ∂E and its interior. We assign to E the canonical orientation, and to ∂E the induced orientation corresponding to a normal vector pointing outwards.

There is then a natural way of constructing a $(n-1)$-form, using the Hodge star operator from Section 4.8. Let η be a one-form on $U \subseteq \mathbb{R}^n$. Then $\star\omega$ is a $(n-1)$-form on $U \subseteq \mathbb{R}^n$, by definition of the Hodge star. So we can rewrite the generalized Stokes' theorem for the one-form η as follows:

$$\int_E d(\star\eta) = \int_{\partial E} \star\eta.$$

This is really the same generalized Stokes' theorem, but instead of writing it in terms of a $(n-1)$-form ω, we write it in terms of a one-form η.

Why would that be of any use? The advantage is that we can easily translate to vector field concepts for all $\mathbb{R}^n$, since we can establish a direct translation between one-forms and vector fields regardless of the dimension.

6.3.3 The divergence theorem in $\mathbb{R}^n$

There is a natural dictionary between one-forms and vector fields in $\mathbb{R}^n$. Let η be a one-form on $U \subseteq \mathbb{R}^n$. We can write:

$$\eta = \sum_{i=1}^n f_i dx_i,$$

where the $f_i : U \to \mathbb{R}$, for $i = 1, \ldots, n$, are smooth functions. We can associate to this one-form the smooth vector field

$$\mathbf{F} = (f_1, f_2, \ldots, f_n)$$

on $U \subseteq \mathbb{R}^n$.

We would like to rewrite our variant of the generalized Stokes' theorem as an integral theorem for the vector field $\mathbf{F}$. Let us first prove a lemma that will enable us to rewrite the left-hand-side of the generalized Stokes' theorem.

Lemma 6.3.1 Rewriting the left-hand-side. *Let $\eta = \sum_{i=1}^n f_i dx_i$ be a one-form on $\mathbb{R}^n$ with associated vector field $\mathbf{F} = (f_1, \ldots, f_n)$. Then*

$$d(\star\eta) = (\boldsymbol{\nabla} \cdot \mathbf{F})\ dx_1 \wedge \ldots \wedge dx_n,$$

and hence we can write

$$\int_E d(\star\eta) = \underbrace{\int \cdots \int_E}_{n\ times} (\boldsymbol{\nabla} \cdot \mathbf{F})\ dV_n,$$

which is a multiple ("n-tuple") integral over the closed bounded region $E \subset \mathbb{R}^n$.

Proof. By definition of the Hodge star, we have:

$$\star\eta = \sum_{i=1}^n (-1)^{i+1} f_i dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where the hat notation means that we take the wedge product of all dx_j's except the dx_i. Calculating the exterior derivative, we get:

$$\begin{aligned} d(\star\eta) &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

$$=(\mathbf{\nabla} \cdot \mathbf{F}) dx_1 \wedge \cdots \wedge dx_n.$$

■

As for the right-hand-side of the generalized Stokes' theorem, we need to rewrite the integral $\int_{\partial E} \star\eta$ in terms of vector calculus objects. ∂E is a closed $(n-1)$-dimensional space in $\mathbb{R}^n$. We can think of it as a parametric space $\alpha : D \to \mathbb{R}^n$ for some closed bounded region $D \subset \mathbb{R}^{n-1}$, like we did for parametric curves in $\mathbb{R}^2$ and parametric surfaces in $\mathbb{R}^3$. In this we case, we can define the integral by pulling back using the parametrization. We claim that the following lemma holds:

Lemma 6.3.2 Rewriting the right-hand-side. *If the boundary space ∂E is realized as a parametric space $\alpha : D \to \mathbb{R}^n$,*

$$\int_{\partial E} \star\eta = \underbrace{\int \cdots \int_D}_{(n-1)\ times} (\mathbf{F} \cdot \mathbf{n}) dV_{n-1},$$

which is a multiple ("$(n-1)$-tuple") integral over the closed bounded region $D \subset \mathbb{R}^{n-1}$. Here, $\mathbf{n}$ is the normal vector to ∂E pointing outward.[1]

Proof. We will not prove this statement in general; we will only prove it for parametric curves and surfaces. In fact, for parametric surfaces, this is basically the statement that was already proven in Corollary 5.6.5; indeed, what we have in this case is a surface integral in $\mathbb{R}^3$, and because by Table 4.1.11 we know that the vector field associated to the two-form $\star\eta$ is the same as the vector field associated to the one-form η, the result of Corollary 5.6.5 still holds here.

Let us then show that it holds for parametric curves in $\mathbb{R}^2$. In this case, $\eta = f\ dx + g\ dy$, with associated vector field $\mathbf{F} = (f, g)$, and $\star\eta = f\ dy - g\ dx$. Let $\alpha : [a, b] \to \mathbb{R}^2$ be a parametric curve representing the boundary curve ∂E. Thus we have:

$$\int_{\partial E} \star\eta = \int_{\alpha} \star\eta = \int_{[a,b]} \alpha^*(\star\eta).$$

If we write $\alpha(t) = (x(t), y(t))$, the pullback is

$$\alpha^*(\star\eta) = \big(f(\alpha(t))y'(t) - g(\alpha(t))x'(t)\big)\ dt.$$

Now, the tangent vector to the parametric curve is

$$\mathbf{T}(t) = (x'(t), y'(t)).$$

The outward pointing normal vector is then

$$\mathbf{n}(t) = (y'(t), -x'(t)),$$

as the two vectors must be orthogonal, and the overall sign of the normal vector is fixed by requiring the it points outwards. We thus see that we can write

$$\alpha^*(\star\eta) = (\mathbf{F} \cdot \mathbf{n})\ dt,$$

and

$$\int_{\partial E} \star\eta = \int_{[a,b]} (\mathbf{F} \cdot \mathbf{n}) dt.$$

■

Putting this together, we see that our variant of the generalized Stokes' theorem gives rise to a generalization of the divergence theorem of the previous section that now holds in any dimension.

Theorem 6.3.3 Divergence theorem in $\mathbb{R}^n$. *Let $\mathbf{F}$ be a vector field on $U \subseteq \mathbb{R}^n$. Let $\alpha : D \to \mathbb{R}^n$ be a parametric $(n-1)$-dimensional space, whose image $\partial E = \alpha(D) \subset U$ is closed. Let $E \subset \mathbb{R}^n$ be the region consisting of the closed surface ∂E and its interior. Let $\mathbf{n}$ be the normal vector to ∂E pointing outwards. Then*

$$\underbrace{\int \cdots \int_E}_{n \text{ times}} (\boldsymbol{\nabla} \cdot \mathbf{F}) dV_n = \underbrace{\int \cdots \int_D}_{(n-1) \text{ times}} (\mathbf{F} \cdot \mathbf{n}) dV_{n-1},$$

where both sides should be understood as multiple integrals over the corresponding regions.

To be precise, we need to specify what normal vector $\mathbf{n}$ we are using here. In $\mathbb{R}^3$, we take the normal vector $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ induced by a parametrization of the surface ∂E (with the right

[1] To be precise, we would need to specify what normal vector we are considering here, since it is not of unit length. As we will see, in $\mathbb{R}^3$ the normal vector is the usual one $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ induced by the parametrization of the surface, while in $\mathbb{R}^2$ it is the normal vector that has the same norm as the tangent vector $\mathbf{T}$ to the parametric curve.

orientation); in $\mathbb{R}^2$, we take the normal vector $\mathbf{n} = (y'(t), -x'(t))$ in terms of a parmaetrization $\alpha(t) = (x(t), y(t))$ of the curve (with the right orientation), which has the same norm as the tangent vector to the parametric curve, that is, $|\mathbf{n}| = |\mathbf{T}| = \sqrt{(x'(t))^2 + (y'(t))^2}$.

Remark 6.3.4 In some textbooks, the divergence theorem in $\mathbb{R}^2$ is simply called "another form of Green's theorem". The reason is that it actually follows directly from Green's theorem. Recall that, given a vector field $\mathbf{F} = (f, g)$ in $\mathbb{R}^2$, Green's theorem states that

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{e}_3 \; dA = \int_{\partial D} (\mathbf{F} \cdot \mathbf{T}) \; dt.$$

The left-hand-side can be rewritten explicitly as

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{e}_3 \; dA = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \; dA,$$

while the right-hand-side can be rewriten as

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{T}) \; dt = \int_{\partial D} \left(f(\alpha(t)) x'(t) + g(\alpha(t)) y'(t) \right) \; dt,$$

where $\alpha(t) = (x(t), y(t))$ is a parametrization of the curve ∂D.

Now if we consider a new vector field $\mathbf{G} = (-g, f)$, Green's theorem applied to $\mathbf{G}$ is the statement that

$$\iint_D (\nabla \times \mathbf{G}) \cdot \mathbf{e}_3 \; dA = \int_{\partial D} (\mathbf{G} \cdot \mathbf{T}) \; dt,$$

which becomes, once written out explicitly,

$$\iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \; dA = \int_{\partial D} \left(f(\alpha(t)) y'(t) - g(\alpha(t)) x'(t) \right) \; dt.$$

But if we rewrite this expression in terms of the original vector field $\mathbf{F} = (f, g)$, we get

$$\iint_D (\nabla \cdot \mathbf{F}) \; dA = \int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) \; dt,$$

which is the divergence theorem in $\mathbb{R}^2$ for $\mathbf{F}$!

So Green's theorem and the divergence theorem in $\mathbb{R}^2$ are really equivalent. But we prefer to call the later the divergence theorem in $\mathbb{R}^2$ as it is the special case of the general divergence theorem in $\mathbb{R}^n$.

Remark 6.3.5 Comparing Green's theorem and the divergence theorem in $\mathbb{R}^2$, it is interesting to note that the curl is related the tangential component of the vector field, while the divergence is related to the normal component. This is not a coincidence; if you recall from Section 4.5, the curl and divergence of vector fields are given a physical interpretation in terms of a moving fluid. The curl concerns whether a small sphere immersed in the fluid will rotate due to the fluid motion -- the rotation will be induced by the tangential component of the velocity field of the fluid on the surface of the sphere. The divergence concerns whether there is more fluid exiting than entering a small sphere immersed in the fluid -- this is mostly influenced by the normal component of the velocity field on the surface of the sphere. In fact, we can make this physical interpretation of the curl and div precise by applying the Stokes' and divergence theorem (respectively) to the small sphere, and take a limit where the volume of the sphere goes to zero. See for instance Section 4.4.1 in CLP 4 for this detailed calculation.

6.4 Applications of the divergence theorem

In this section we study a few applications of the divergence theorem in $\mathbb{R}^n$.

Objectives

You should be able to:

- Use the divergence theorem in the context of applications in science.

6.4.1 The divergence theorem in $\mathbb{R}^3$ and the heat equation

Our first application concerns heat flow in $\mathbb{R}^3$. First, we recall the divergence theorem in $\mathbb{R}^3$. Let $\mathbf{F}$ be a smooth vector field, ∂E a closed surface with normal vector pointing outward, and E the solid region consisting of ∂E and its interior with canonical orientation. The divergence theorem in $\mathbb{R}^3$ is the statement that

$$\iiint_E (\boldsymbol{\nabla} \cdot \mathbf{F})\ dV = \iint_{\partial E} (\mathbf{F} \cdot \mathbf{n}) dA.$$

We consider the case where the vector field $\mathbf{F}$ is the heat flow. Recall the context from Subsection 5.9.2. Suppose that the temperature at a point (x, y, z) in an object (or substance) is given by the function $T(x, y, z)$. The heat flow is given the gradient of the temperature function, rescaled by a constant K known as the conductivity of the substance:

$$\mathbf{F} = -K\boldsymbol{\nabla} T.$$

Now consider any closed surface ∂E, with the surface ∂E and its interior within the object. The amount of heat flowing across the surface ∂E is given by the flux of the vector field $\mathbf{F}$ across ∂E:

$$\iint_{\partial E} (\mathbf{F} \cdot \mathbf{n}) dA = -K \iint_{\partial E} (\boldsymbol{\nabla} T \cdot \mathbf{n})\, dA.$$

If we are interested in the amount of heat entering the solid region E (instead of flowing across the surface in the outward direction), we change the sign of the flux integral. Then, by the divergence theorem, the amount of heat entering the solid region E can be rewritten as a volume integral:

$$\begin{aligned} K \iint_{\partial E} (\boldsymbol{\nabla} T \cdot \mathbf{n})\, dA =& K \iiint_E (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T)\ dV \\ =& K \iiint_E \nabla^2 T\ dV, \end{aligned}$$

where $\nabla^2 T$ is the Laplacian of the temperature function T.

Now we want to consider the situation where the temperature function T is also changing in time. So we think of T as a function of four variables $T = T(x, y, z, t)$. But t is just a "spectator variable" here; we still consider the operator ∇^2 in $\mathbb{R}^3$, in terms of the variables (x, y, z). So we can go through all the steps above, and we obtain that the amount of heat entering the solid region E during a small (infinitesimal) amount of time dt is

$$K dt \iiint_E \nabla^2 T \; dV.$$

Here the Laplacian is only in the variables (x, y, z), that is, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

To proceed further, we need a little bit of physics. It is known in physics that the amount of heat energy required to raise the temperature of an object by ΔT is given by $CM\Delta T$, where M is the mass of the object and C is constant known as the "specific heat" of the material. Now consider the object consisting of the solid region E. In a small (infinitesimal) amount of time dt, the temperature changes by $\frac{\partial T(x,y,z,t)}{\partial t} \; dt$. If we consider an infinitesimal volume element dV in E, and $\rho(x, y, z)$ is the mass density of the solid region, then the mass of the volume element is ρdV. Thus the heat energy required to change the temperature of the object in the time interval dt is

$$C\rho \frac{\partial T}{\partial t} \; dV dt.$$

We then sum over all volume elements, i.e. integrate over E, to get that the total heat energy required to change the temperature during the time interval dt is

$$C dt \iiint_E \rho \frac{\partial T}{\partial t} \; dV.$$

Assuming that the object is not creating heat energy itself, this heat energy should be equal to the amount of heat entering the solid region E through the boundary surface ∂E during the time interval dt, which is what we calculated previously. We thus obtain the equality:

$$C dt \iiint_E \rho \frac{\partial T}{\partial t} \; dV = K dt \iiint_E \nabla^2 T \; dV.$$

We can cancel the time interval dt on both sides. Rewriting both terms on the same side of the equality, we get:

$$\iiint_E \left(K\nabla^2 T - C\rho \frac{\partial T}{\partial t} \right) \; dV = 0.$$

But this must be true for all solid regions E within the object, and for all times t. From this we can conclude that the integrand must be identically zero:

$$K\nabla^2 T = C\rho \frac{\partial T}{\partial t}.$$

This equation is generally rewritten as

$$\frac{\partial T(x, y, z, t)}{\partial t} = \alpha \nabla^2 T(x, y, z, t),$$

where $\alpha = \frac{K}{C\rho}$ is called the "thermal diffusivity", and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

This equation is very famous: it is known as the **heat equation**. As mentioned in the Wikipedia page on "Heat equation",

> As the prototypical parabolic partial differential equation, the heat equation is among the most widely studied topics in pure mathematics, and its analysis is regarded as fundamental to the broader field of partial differential equations.

The importance of the heat equation goes beyond physics and heat flow. It has a wide range of applications, from the physics of heat flow of course, to probability theory, to financial mathematics, to quantum mechanics, to image analysis in computer science. A generalization of the heat equation is also behind the famous proof of the Poincare conjecture by Pereleman in 2003 (the only Millenium Prize Problem that has been solved so far). I encourage you to have a look at the wikipedia page on the heat equation!

6.4.2 The divergence theorem in $\mathbb{R}^n$ and Green's first and second identities

We now consider the divergence theorem in $\mathbb{R}^n$. Let $\mathbf{F}$ be a vector field, ∂E a closed $(n-1)$-dimensional subspace with normal vector pointing outward, and E the region of $\mathbb{R}^n$ consisting of ∂E and its interior with canonical orientation. The divergence theorem in $\mathbb{R}^n$ is the statement that

$$\underbrace{\int\cdots\int_E}_{n \text{ times}} (\boldsymbol{\nabla}\cdot\mathbf{F})dV_n = \underbrace{\int\cdots\int_{\partial E}}_{(n-1) \text{ times}} (\mathbf{F}\cdot\mathbf{n})dV_{n-1}.$$

Using this theorem, we can prove the following two identities, known as Green's first and second identities.

Lemma 6.4.1 Green's first identity. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions with continuous partial derivatives. Let E and ∂E be as above. Then*

$$\underbrace{\int\cdots\int_E}_{n \textit{ times}} f\nabla^2 g \; dV_n = \underbrace{\int\cdots\int_{\partial E}}_{(n-1) \textit{ times}} f\boldsymbol{\nabla} g\cdot\mathbf{n}\; dV_{n-1} - \underbrace{\int\cdots\int_E}_{n \textit{ times}} (\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g)\; dV_n.$$

Proof. We consider the divergence theorem in $\mathbb{R}^n$ with vector field $\mathbf{F} = f\boldsymbol{\nabla} g$. By the third identity in Lemma 4.4.9, we know that

$$\boldsymbol{\nabla}\cdot(f\mathbf{F}) = (\boldsymbol{\nabla} f)\cdot\mathbf{F} + f\boldsymbol{\nabla}\cdot\mathbf{F}.$$

Thus

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g) =& \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g + f\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} g\\ =& \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g + f\nabla^2 g.\end{aligned}$$

Therefore, the divergence theorem applied to $\mathbf{F} = f\boldsymbol{\nabla} g$ becomes:

$$\underbrace{\int\cdots\int_E}_{n \text{ times}} \left(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g + f\nabla^2 g\right)\; dV_n = \underbrace{\int\cdots\int_{\partial E}}_{(n-1) \text{ times}} f\boldsymbol{\nabla} g\cdot\mathbf{n}\; dV_{n-1},$$

which is the statement of Green's first identity. ∎

This may not be obvious at first, but Green's first identity is essentially the equivalent of integration by parts in higher dimension. Basically, integration by parts can be written as

$$\int_a^b f\ dg = fg\Big|_a^b - \int_a^b g\ df.$$

Green's first identity generalizes this statement for the n-tuple integral of the function $f\nabla^2 g$ over a closed bounded region $E \subset \mathbb{R}^n$.

Lemma 6.4.2 Green's second identity. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions with continuous partial derivatives. Let E and ∂E be as above. Then*

$$\underbrace{\int \cdots \int_E}_{n\ times} (f\nabla^2 g - g\nabla^2 f)\ dV_n = \underbrace{\int \cdots \int_{\partial E}}_{(n-1)\ times} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f)\cdot \mathbf{n}\ dV_{n-1}.$$

Proof. Green's second identity follows from the first identity. Using the first identity, we know that

$$\begin{aligned}\underbrace{\int \cdots \int_E}_{n \text{ times}} (f\nabla^2 g - g\nabla^2 f)\ dV_n = &\underbrace{\int \cdots \int_{\partial E}}_{(n-1) \text{ times}} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f)\cdot \mathbf{n}\ dV_{n-1}\\ &- \underbrace{\int \cdots \int_E}_{n \text{ times}} (\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g - \boldsymbol{\nabla} g\cdot\boldsymbol{\nabla} f)\ dV_n.\end{aligned}$$

But $\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g = \boldsymbol{\nabla} g\cdot\boldsymbol{\nabla} f$, and hence the last term vanishes. The result is Green's second identity. ■

Green's identities are quite useful in mathematics. There is in fact also a third Green's identity, but it is beyond the scope of this class. Have a look at the Wikipedia page on "Green's identities" if you are interested!

6.4.3 Exercises

1. Recall that a function $g : U \to \mathbb{R}$ with $U \subseteq \mathbb{R}^n$ is **harmonic** on U if it is a solution to the Laplace equation, that is, $\nabla^2 g = 0$ on U, where $\nabla^2 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Use Green's first identity to show that if g is harmonic on U, and $E \subset U$ (with E and ∂E as usual), then

$$\underbrace{\int \cdots \int_{\partial E}}_{(n-1) \text{ times}} \boldsymbol{\nabla} g\cdot \mathbf{n}\ dV_{n-1} = 0.$$

Solution. We consider Green's identity with the constant function $f = 1$. It reads:

$$\underbrace{\int \cdots \int_E}_{n \text{ times}} \nabla^2 g\ dV_n = \underbrace{\int \cdots \int_{\partial E}}_{(n-1) \text{ times}} \boldsymbol{\nabla} g\cdot \mathbf{n}\ dV_{n-1} - \underbrace{\int \cdots \int_E}_{n \text{ times}} (\boldsymbol{\nabla}(1)\cdot\boldsymbol{\nabla} g)\ dV_n.$$

But $\boldsymbol{\nabla}(1) = 0$, since the gradient of a constant function necessarily vanishes. Furthermore, since we assume that g is harmonic, we know that $\nabla^2 g = 0$. Therefore we conclude that

$$\underbrace{\int \cdots \int_{\partial E}}_{(n-1)\text{ times}} \boldsymbol{\nabla} g \cdot \mathbf{n}\, dV_{n-1} = 0.$$

2. As in the previous exercise, let g be a harmonic function on $U \subseteq \mathbb{R}^n$, with $E \subset U$. Use Green's first identity to show that if $g = 0$ on the boundary space ∂E (with E and ∂E as usual), then

$$\underbrace{\int \cdots \int_{E}}_{n\text{ times}} |\boldsymbol{\nabla} g|^2\, dV_n = 0.$$

Solution. We consider Green's first identity again, but now with $f = g$. It reads:

$$\underbrace{\int \cdots \int_{E}}_{n\text{ times}} g\nabla^2 g\, dV_n = \underbrace{\int \cdots \int_{\partial E}}_{(n-1)\text{ times}} g\boldsymbol{\nabla} g \cdot \mathbf{n}\, dV_{n-1} - \underbrace{\int \cdots \int_{E}}_{n\text{ times}} (\boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} g)\, dV_n.$$

We assume that g is harmonic, that is, $\nabla^2 g = 0$. Furthermore, we assume that the function g vanishes on the boundary surface ∂E, therefore the integral

$$\underbrace{\int \cdots \int_{\partial E}}_{(n-1)\text{ times}} g\boldsymbol{\nabla} g \cdot \mathbf{n}\, dV_{n-1}$$

vanishes, since the integrand is identically zero on the surface ∂E over which we are integrating. As a result, Green's first identity becomes

$$\underbrace{\int \cdots \int_{E}}_{n\text{ times}} (\boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} g)\, dV_n = 0.$$

But $\boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} g = |\boldsymbol{\nabla}(g)|^2$, and we obtain

$$\underbrace{\int \cdots \int_{E}}_{n\text{ times}} |\boldsymbol{\nabla} g|^2\, dV_n = 0.$$

6.5 Integral theorems: when to use what

This is not really an independent section (or an independent lecture), but just a brief summary of the typical usages of the various integral theorems that we have seen so far.

Objectives

You should be able to:

- Determine which integral theorem may be useful to evaluate certain type of line and surface integrals.

We have seen five integral theorems so far, all particular cases of the generalized Stokes' theorem:

1. The Fundamental Theorem of calculus;
2. The Fundamental Theorem of line integrals;
3. Green's theorem;
4. Stokes' theorem;
5. The divergence theorem.

One of the main difficulties with the integral theorems of calculus is to determine which theorem may be helpful in a given situation. In this section I list a few typical situations for which integral theorems may be useful, highlighting the main applications of the integral theorems. You can use this list as a rule of thumb.

Strategy 6.5.1 Integral theorems: when to use what.

- You want to evaluate a line integral along a curve in $\mathbb{R}^n$ for an exact one-form (or the gradient of a function): **Fundamental Theorem of line integrals** (Section 3.4).
- You want to evaluate a line integral along a closed curve in $\mathbb{R}^2$: **Green's theorem** (Section 5.7).
- You want to evaluate a line integral along a closed curve in $\mathbb{R}^3$: **Stokes' theorem** (Section 5.8).
- You want to evaluate a surface integral along a surface in $\mathbb{R}^3$ for an exact two-form (or the curl of a vector field): **Stokes' theorem** (Section 5.8).
- You want to evaluate a surface integral along a closed surface in $\mathbb{R}^3$: **Divergence theorem in** $\mathbb{R}^3$ (Section 6.2).

Chapter 7

Unoriented line and surface integrals

7.1 Unoriented line integrals

We define unoriented line integrals of functions along parametric curves. As a particular case, we study how to calculate the arc length of a parametric curve.

Objectives

You should be able to:

- Determine the arc length of a parametrized curve in $\mathbb{R}^n$ using an unoriented line integral.
- Evaluate the unoriented integral of a function along a parametrized curve in $\mathbb{R}^n$.

7.1.1 Unoriented line integrals

In this course we developed a theory of integration along curves and surfaces using differential forms. By construction, our theory was oriented, as integrals of differential forms naturally depend on a choice of orientation on the space over which we are integrating.

However, not all integrals should be oriented. Sometimes we want to calculate a quantity associated to a curve or a surface that should not depend on a choice of orientation. Typical examples would be the length of a curve or the area of a surface: such quantities should not depend on a choice of orientation. As integrals of differential forms are naturally oriented, it follows that integrals calculating arc lengths or surface areas cannot be represented as integrals of differential forms. We need to study unoriented line and surface integrals. In this section we look at unoriented line integrals.

Before we define the concept of unoriented line integral of a function along a parametric curve, let us review how we defined oriented line integrals. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with $\alpha(t) = (x_1(t), \dots, x_n(t))$. We defined the oriented line integral of a one-form ω along the parametric curve α via pullback (Definition 3.3.2). In terms of the vector field $\mathbf{F} = (f_1, \dots, f_n)$ associated to the one-form ω, by evaluating the pullback the oriented line integral can be rewritten as (Lemma 3.3.7):

$$\int_a^b \left(\mathbf{F}(\alpha(t)) \cdot \mathbf{T}\right)\, dt,$$

where $\mathbf{T}$ is the tangent vector

$$\mathbf{T}(t) = (x_1'(t), \ldots, x_n'(t)).$$

From the point of view of vector fields, the orientation of the integral is encapsulated in the choice of tangent vector $\mathbf{T}$. However, the way it is formulated, the tangent vector includes more information than just the orientation, as it also has a non-trivial norm specified by the parametrization. To isolate the oriented nature of the integral, we normalize the tangent vector, and define the unit tangent vector

$$\hat{\mathbf{T}} = \frac{\mathbf{T}}{|\mathbf{T}|}, \qquad |\mathbf{T}| = \sqrt{(x_1'(t))^2 + \ldots + (x_n'(t))^2}.$$

We can then rewrite the oriented line integral as

$$\int_a^b \left(\mathbf{F}(\alpha(t)) \cdot \hat{\mathbf{T}}\right) |\mathbf{T}| \ dt = \int_a^b \left(\mathbf{F}(\alpha(t)) \cdot \hat{\mathbf{T}}\right) \ ds,$$

where we defined the "line element"

$$ds = |\mathbf{T}| \ dt = \sqrt{(x_1'(t))^2 + \ldots + (x_n'(t))^2} \ dt.$$

With this formulation, we see that the choice of orientation is completely encapsulated in the expression $\mathbf{F}(\alpha(t)) \cdot \hat{\mathbf{T}}(t)$, which is function of t which depends on the choice of direction on the parametric curve.

We can now see how unoriented line integrals can be naturally defined: we simply replace the function $\mathbf{F}(\alpha(t)) \cdot \hat{\mathbf{T}}(t)$, constructed out of a vector field and a choice of orientation on the curve, by an arbitrary function $f(\alpha(t))$ that does not depend on a choice of orientation. This leads to the following definition.

Definition 7.1.1 Unoriented line integrals. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with image curve $C = \alpha([a, b]) \subset \mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a continuous function. We define the **unoriented line integral of f along the curve** C to be

$$\int_C f \ ds = \int_a^b f(\alpha(t))|\mathbf{T}(t)| \ dt = \int_a^b f(\alpha(t))\sqrt{(x_1'(t))^2 + \ldots + (x_n'(t))^2} \ dt.$$

◊

A similar calculation as in the proof of Lemma 3.3.5 shows that unoriented line integrals are invariant under reparametrizations, regardless of whether the reparametrization preserves the orientation or not (the integrals are unoriented). What this means is that the line integral does not depend on the choice of parametrization, but only on the image curve C. This is why we wrote

$$\int_C f \ ds$$

for the unoriented line integral of the function f along the curve $C \subset \mathbb{R}^n$, without specifying the parametrization α, since the integral is independent of the choice of parametrization.

Remark 7.1.2 We note that even though the notation "ds" is similar to the notation we used for one-forms, the line element is not a one-form. For instance,

$$\int_{C_+} ds = \int_{C_-} ds,$$

i.e. the integral remains the same if we change the orientation of the curve, which would not be the case if ds was a one-form.

Example 7.1.3 An example of an unoriented line integral. Evaluate the unoriented line integral

$$\int_C xy^6 \ ds,$$

where C is the right half of the circle $x^2 + y^2 = 4$.

First, we parametrize the curve as $\alpha : [-\pi/2, \pi/2] \to \mathbb{R}^2$ with

$$\alpha(\theta) = (2\cos(\theta), 2\sin(\theta)).$$

As $\theta \in [-\pi/2, \pi/2]$, we are parametrizing the right half of the circle, as required. We do not need to check here whether the parametrization induces the right orientation on the curve, as we do not care about the orientation whatsoever: the integral is unoriented.

To evaluate the unoriented line integral, we need the line element ds. We calculate:

$$\begin{aligned} ds =& \sqrt{(x'(\theta))^2 + (y'(\theta))^2} \\ =& \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)} \\ =& 2. \end{aligned}$$

Using this parametrization, the unoriented line integral becomes:

$$\begin{aligned} \int_C xy^6 \ ds =& \int_{-\pi/2}^{\pi/2} (2\cos(\theta))(2\sin(\theta))^6 (2) \ d\theta \\ =& 256 \int_{-\pi/2}^{\pi/2} \cos(\theta)\sin^6(\theta) \ d\theta \\ =& 256 \int_{-1}^{1} u^6 \ du \\ =& \frac{512}{7}. \end{aligned}$$

Note that we used the substitution $u = \sin(\theta)$ to evaluate the trigonometric integral. □

7.1.2 Arc length of a curve

A particularly important example of an unoriented line integral calculates the arc length of a curve C. This is the most trivial example, where we choose the function that we are integrating to be the constant function $f = 1$. More precisely:

Definition 7.1.4 Arc length of a curve. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametric curve, with image curve $C = \alpha([a,b]) \subset \mathbb{R}^n$. The **arc length of** C is given by the unoriented line integral

$$\int_C ds = \int_a^b \sqrt{(x_1'(t))^2 + \ldots + (x_n'(t))^2}\ dt.$$

$\Diamond$

You may have seen this formula before for the arc length, at least in $\mathbb{R}^2$ or $\mathbb{R}^3$. It is straightforward to justify that it calculates the arc length of the curve, using the standard slicing argument from integral calculus. Consider a small curve segment between two points $P(\alpha(t))$ and $Q(\alpha(t+dt))$. The length ds of this curve segment can be approximated by the length of the line joining the two points, which can be written as

$$ds \simeq |\alpha(t+dt) - \alpha(t)| \simeq |\alpha'(t)|\, dt,$$

where on the right-hand-side we kept only terms of first-order in dt. As $\alpha'(t) = \mathbf{T}(t)$, we recover the formula above for the line element. Finally, we sum over line elements and take the limit of an infinite number of line element of infinitesimal size, which turns the sum into the definite integral

$$\int_a^b |\mathbf{T}(t)|\ dt = \int_a^b \sqrt{(x_1'(t))^2 + \ldots + (x_n'(t))^2}\ dt.$$

Note that this also justifies our definition of unoriented line integrals in general above; it is constructed via the same slicing process, but where we also introduce a function f evaluated the point $\alpha(t)$ where we calculate the line element. We then sum over slices and take the limit of infinite number of infinitesimal slices as usual, and the integral of the function over the parametric curve becomes Definition 7.1.1.

Example 7.1.5 Calculating the arc length of a parametric curve. Find the length of the parametric curve $\alpha : [0,2] \to \mathbb{R}^3$ with

$$\alpha(t) = (1, t^2, t^3).$$

We first calculate the line element ds:

$$\begin{aligned} ds =& \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt \\ =& \sqrt{4t^2 + 9t^4}\ dt \\ =& t\sqrt{4 + 9t^2}\ dt, \end{aligned}$$

where in the last line we used $\sqrt{t^2} = t$ since we know that $t \in [0,2]$ and hence it is positive. The arc length is thus given by

$$\begin{aligned} \int_C ds =& \int_0^2 t\sqrt{4+9t^2}\ dt \\ =& \frac{1}{18}\int_4^{40} \sqrt{u}\ du \\ =& \frac{1}{27}(40^{3/2} - 4^{3/2}) \\ =& \frac{8}{27}(10^{3/2} - 1), \end{aligned}$$

where we used the substitution $u = 4 + 9t^2$, $du = 18t\ dt$. $\square$

7.1.3 Exercises

1. Find the arc length of the circular helix $\alpha : [0, 3] \to \mathbb{R}^3$ with

$$\alpha(t) = (t, 2\cos(t), 2\sin(t)).$$

Solution. To find the arc length, we first calculate the line element ds. The tangent vector is

$$\mathbf{T}(t) = (1, -2\sin(t), 2\cos(t)).$$

Its norm is

$$|\mathbf{T}(t)| = \sqrt{1 + 4\sin^2(t) + 4\cos(t)} = \sqrt{5}.$$

Thus the line element is

$$ds = \sqrt{5}\ dt.$$

We then calculate the arc length:

$$\begin{aligned}\int_\alpha ds =& \int_0^3 \sqrt{5}\ dt \\ =& 3\sqrt{5}.\end{aligned}$$

2. Show that the arc length of the curve C at the intersection of the surfaces $x^2 = 2y$ and $3z = xy$ between the origin and the point $(6, 18, 36)$ is the answer to the ultimate question of life, the universe, and everything.

Solution. We first parametrize the curve as $\alpha : [0, 6] \to \mathbb{R}^3$ with

$$\alpha(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{6}\right).$$

The tangent vector is

$$\mathbf{T}(t) = \left(1, t, \frac{t^2}{2}\right).$$

Its norm is

$$|\mathbf{T}(t)| = \sqrt{1 + t^2 + \frac{t^4}{4}} = \sqrt{\left(1 + \frac{t^2}{2}\right)^2} = 1 + \frac{t^2}{2},$$

since $1 + \frac{t^2}{2} > 0$. The line element is then

$$ds = \left(1 + \frac{t^2}{2}\right)\ dt,$$

and the arc length is

$$\begin{aligned}\int_C ds =& \int_0^6 \left(1 + \frac{t^2}{2}\right)\ dt \\ =& 6 + \frac{6^3}{6}\end{aligned}$$

$$=42,$$

which is of course the answer to the ultimate question of life, the univers, and everything! :-)

3. Evaluate the unoriented line integral

$$\int_C (xz + e^{-y})\ ds,$$

where C is the line segment between the origin and the point $(1, 2, 3)$.

Solution. We parametrize C as $\alpha : [0, 1] \to \mathbb{R}^3$ with

$$\alpha(t) = (t, 2t, 3t).$$

The tangent vector is

$$\mathbf{T}(t) = (1, 2, 3),$$

with norm

$$|\mathbf{T}(t)| = \sqrt{1 + 2^2 + 3^2} = \sqrt{14}.$$

The line element is

$$ds = \sqrt{14}\ dt,$$

and the unoriented line integral can be evaluated:

$$\begin{aligned}\int_C (xz + e^{-y})\ ds = & \int_0^1 ((t)(3t) + e^{-2t})\sqrt{14}\ dt \\ =& \sqrt{14} \int_0^1 (3t^2 + e^{-2t})\ dt \\ =& \sqrt{14} \left(1 - \frac{e^{-2}}{2} + \frac{1}{2}\right) \\ =& \frac{\sqrt{14}}{2}(3 - e^{-2}).\end{aligned}$$

4. Evaluate the unoriented line integral

$$\int_C x^3 y\ ds,$$

where C is the circular helix parametrized by $\alpha : [0, \pi/2] \to \mathbb{R}^3$ with

$$\alpha(t) = (\cos(2t), \sin(2t), t).$$

Solution. The tangent vector to the parametric curve is

$$\mathbf{T}(t) = (-2\sin(2t), 2\cos(2t), 1),$$

with norm

$$|\mathbf{T}(t)| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 1} = \sqrt{5}.$$

The line element is

$$ds = \sqrt{5}\ dt.$$

The unoriented line integral becomes:

$$\begin{aligned}\int_C x^3 y\ ds =& \sqrt{5}\int_0^{\pi/2} \cos^3(2t)\sin(2t)\ dt\\ =& -\frac{\sqrt{5}}{2}\int_1^{-1} u^3\ du\\ =& 0,\end{aligned}$$

where we did the substitution $u = \cos(2t)$.

5. In single-variable calculus, you saw that the length of the curve $y = f(x)$ in $\mathbb{R}^2$, with $x \in [a, b]$, is given by the definite integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

Show that this is consistent with our definition of arc length in this section.

Solution. From our point of view, we realize the curve as the parametric curve $\alpha : [a, b] \to \mathbb{R}^2$ with

$$\alpha(t) = (t, f(t)).$$

Then the tangent vector is

$$\mathbf{T}(t) = (1, f'(t)),$$

with norm

$$|\mathbf{T}(t)| = \sqrt{1 + (f'(t))^2}.$$

So our arc length formula is

$$\int_C ds = \int_a^b |\mathbf{T}(t)|\ dt = \int_a^b \sqrt{1 + (f'(t))^2}\ dt,$$

which is indeed the formula that you obtained in single-variable calculus.

7.2 Unoriented surface integrals

We define unoriented surface integrals of functions along parametric surfaces in $\mathbb{R}^3$. As a special case, we study how to calculate the surface area of a surface in $\mathbb{R}^3$.

Objectives

You should be able to:

- Determine the surface area of a parametrized surface in $\mathbb{R}^3$ using an unoriented surface integral.
- Evaluate the unoriented integral of a function along a parametrized surface in $\mathbb{R}^3$.

7.2.1 Unoriented surface integrals

In the previous section we defined unoriented line integrals over parametric curves in $\mathbb{R}^n$. Those cannot be defined as integrals of differential forms, as the integrals are independent of a choice of orientation on the curve. We can similarly define unoriented surface integrals, to which we now turn to. Those will be useful to calculate quantities associated to parametric surfaces in $\mathbb{R}^3$ that should not depend on a choice of orientation, such as the surface area of the surface.

We proceed in a way similar to what we did for line integrals, by first recalling our construction of oriented surface integrals using differential forms. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface, with $\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$. We defined the oriented surface integral of a two-form ω along α via pullback to D (Definition 5.6.1). In terms of the vector field $\mathbf{F} = (f_1, f_2, f_3)$ associated to the two-form ω, the surface integral can be written as

$$\iint_D (\mathbf{F}(\alpha(u,v)) \cdot \mathbf{n}) \ dA,$$

where the normal vector $\mathbf{n}$ is

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v, \qquad \mathbf{T}_u = \frac{\partial \alpha}{\partial u}, \qquad \mathbf{T}_v = \frac{\partial \alpha}{\partial v}.$$

The orientation is now encapsulated in the choice of normal vector. But as was the case for oriented line integrals, the normal vector here contains more information than just the orientation, as it is not normalized. We define the unit normal vector as

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|}.$$

We can then rewrite the surface integral as

$$\iint_D (\mathbf{F}(\alpha(u,v)) \cdot \hat{\mathbf{n}}) |\mathbf{T}_u \times \mathbf{T}_v| \ dA = \iint_D (\mathbf{F}(\alpha(u,v)) \cdot \hat{\mathbf{n}}) \ dS,$$

where we defined the "surface element"

$$dS = |\mathbf{T}_u \times \mathbf{T}_v| \ dA.$$

With this rewriting, we can think of the surface integral as integrating the expression $(\mathbf{F}(\alpha(u,v)) \cdot \hat{\mathbf{n}})$, which is a function of the parameters (u,v) that depends on the choice of orientation via the unit normal vector $\hat{\mathbf{n}}$.

Unoriented surface integrals are then naturally defined by replacing the orientation-dependent function $(\mathbf{F}(\alpha(u,v)) \cdot \hat{\mathbf{n}})$ by an arbitrary function $f(\alpha(u,v))$ that does not depend on a choice of orientation on the surface. We obtain the following definition of unoriented surface integrals.

Definition 7.2.1 Unoriented surface integrals. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface, with $\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$, and image surface $S = \alpha(D) \subset \mathbb{R}^3$. The tangent vectors are

$$\mathbf{T}_u = \frac{\partial \alpha}{\partial u}, \qquad \mathbf{T}_v = \frac{\partial \alpha}{\partial v}.$$

Let $f : S \to \mathbb{R}$ be a continuous function. We define the **unoriented integral of f along the surface S** to be

$$\iint_S f\ dS = \iint_D f(\alpha(u,v))|\mathbf{T}_u \times \mathbf{T}_v|\ dA.$$

$\lozenge$

A calculation similar to the proof of Lemma 5.6.3 shows that unoriented surface integrals are invariant under arbitrary reparametrizations of the surface $S \subset \mathbb{R}^3$, regardless of whether the orientation is preserved or not. This is why we wrote

$$\iint_S f\ dS$$

to denote the unoriented surface integral along the surface S, as the integral does not depend on how we parametrize the surface.

Example 7.2.2 An example of an unoriented surface integral. Evaluate the surface integral

$$\iint_S x^2 z\ dS$$

over the cone $z^2 = x^2 + y^2$ above the (x, y)-plane and below the plane $z = 1$.

We first parametrize the surface as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,1], \theta \in [0, 2\pi]\}$$

and

$$\alpha(r,\theta) = (r\cos(\theta), r\sin(\theta), r).$$

To evaluate the surface integral, we need to calculate the surface element dS. The tangent vectors are

$$\mathbf{T}_r = (\cos(\theta), \sin(\theta), 1), \qquad \mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 0).$$

The cross product is (we note here that the order does not matter, as the integral is unoriented; we will calculate the norm of the cross product afterwards, which is the same regardless of whether we take $\mathbf{T}_r \times \mathbf{T}_\theta$ or $\mathbf{T}_\theta \times \mathbf{T}_r$):

$$\mathbf{T}_r \times \mathbf{T}_\theta = (-r\cos(\theta), -r\sin(\theta), r).$$

Its norm is

$$|\mathbf{T}_r \times \mathbf{T}_\theta| = \sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta) + r^2} = \sqrt{2}r,$$

where we used the fact that $\sqrt{r^2} = r$ since $r \in [0, 1]$ and hence is positive.

The unoriented surface integral then becomes

$$\begin{aligned}\iint_S x^2 z\ dS =& \iint_D (r\cos(\theta))^2(r)(\sqrt{2}r)\ dA\\ =& \sqrt{2}\int_0^1\int_0^{2\pi} r^4\cos^2(\theta)d\theta dr\\ =& \sqrt{2}\pi\int_0^1 r^4\ dr\\ =& \frac{\sqrt{2}\pi}{5}.\end{aligned}$$

$\square$

7.2.2 Surface area of a parametric surface in $\mathbb{R}^3$

Just as evaluating the unoriented line integral of the constant function $f = 1$ gave the arc length of the parametric curve, evaluating the unoriented surface integral of the constant function $f = 1$ gives the surface area of the parametric surface.

Definition 7.2.3 Surface area of a parametric surface in $\mathbb{R}^3$. Let $\alpha : D \to \mathbb{R}^3$ be a parametric surface, with $\alpha(u, v) = (x(u, v), y(u, v), z(u, v))$, and image surface $S = \alpha(D) \subset \mathbb{R}^3$. The **surface area of** S is given by the unoriented surface integral

$$\iint_S dS = \iint_D |\mathbf{T}_u \times \mathbf{T}_v| \, dA.$$

$\diamondsuit$

The justification for this definition of the surface area comes from the standard slicing process of integral calculus, as usual. Consider a small rectangle at position (u, v) within the domain D, and with sides of length du and dv. The area of this small rectangle is $dA = dudv$. The parametrization α maps the small rectangle to a small region dS within the surface S. This region is now curved, and not necessarily rectangular. However, one can approximate its area by replacing it by the parallelogram in the tangent plane at the point $\alpha(u, v)$ spanned by the vectors $du\mathbf{T}_u$ and $dv\mathbf{T}_v$. The area of this parallelogram is $dS = |\mathbf{T}_u \times \mathbf{T}_v| dudv$. Finally, we sum over all small regions within the domain D, and take a limit of an infinite number of infinitesimal regions, which turns the approximate calculation of the surface area into an exact one. The result of this limit process is the double integral

$$\iint_D |\mathbf{T}_u \times \mathbf{T}_v| \, dA.$$

Admittedly, the justification is a little bit hand-wavy here, but it can be made precise.

This also justifies our general construction of unoriented surface integrals in Definition 7.2.1: all that we add to the slicing process is a function on the surface S that we evaluate at each point $\alpha(u, v)$ on S before summing over slices to turn the calculation into a double integral.

Example 7.2.4 Calculating the surface area of a parametric surface. Find the surface area of the part of the plane $x + 2y + z = 1$ that lies within the cylinder $x^2 + y^2 = 4$.

As we know that the region will be within the cylinder $x^2 + y^2 = 4$, we use polar coordinates to parametrize the surface. We write down the parametrization $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 2], \theta \in [0, 2\pi]\}$$

and

$$\alpha(r, \theta) = (r \cos(\theta), r \sin(\theta), 1 - r \cos(\theta) - 2r \sin(\theta)).$$

We need to find the surface element dS. The tangent vectors are

$$\mathbf{T}_r = (\cos(\theta), \sin(\theta), -\cos(\theta) - 2\sin(\theta)), \qquad \mathbf{T}_\theta = (-r \sin(\theta), r \cos(\theta), r \sin(\theta) - 2r \cos(\theta)).$$

The cross product is

$$\mathbf{T}_r \times \mathbf{T}_\theta = (r, 2r, r),$$

whose norm is

$$|\mathbf{T}_r \times \mathbf{T}_\theta| = \sqrt{r^2 + 4r^2 + r^2} = \sqrt{6}r,$$

where we used the fact that $\sqrt{r^2} = r$ since $r \in [0, 2]$ and hence is positive.

Finally, we evaluate the surface integral of dS to get the surface area of S:

$$\begin{aligned}
\iint_S dS &= \iint_D |\mathbf{T}_r \times \mathbf{T}_\theta| \; dA \\
&= \sqrt{6} \int_0^2 \int_0^{2\pi} r \; d\theta dr \\
&= 2\sqrt{6}\pi \int_0^2 r \; dr \\
&= 4\sqrt{6}\pi.
\end{aligned}$$

$\square$

7.2.3 Exercises

1. Find the surface area of the part of the surface $z = 4 - 2x^2 + y$ over the triangle with vertices $(0,0), (2,0), (2,2)$ in the (x,y)-plane.

Solution. First, we notice that the triangle in the (x,y)-plane can be realized as the x-supported region $x \in [0,2]$, $0 \le y \le x$. We can then parametrize the surface as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u,v) \in \mathbb{R}^2 \mid u \in [0,2], 0 \le v \le u\}$$

and

$$\alpha(u,v) = (u, v, 4 - 2u^2 + v).$$

The tangent vectors are

$$\mathbf{T}_u = (1, 0, -4u), \qquad \mathbf{T}_v = (0, 1, 1).$$

The cross product is

$$\mathbf{T}_u \times \mathbf{T}_v = (4u, -1, 1).$$

Its norm is

$$|\mathbf{T}_u \times \mathbf{T}_v| = \sqrt{16u^2 + 1 + 1} = \sqrt{2}\sqrt{8u^2 + 1}.$$

The surface area becomes

$$\begin{aligned} \iint_S dS =& \sqrt{2} \int_0^2 \int_0^u \sqrt{8u^2 + 1}\, dvdu \\ =& \sqrt{2} \int_0^2 u\sqrt{8u^2 + 1}\, du \\ =& \frac{\sqrt{2}}{16} \int_1^{33} \sqrt{t}\, dt \\ =& \frac{\sqrt{2}}{24}(33^{3/2} - 1) \\ =& \frac{11\sqrt{66}}{8} - \frac{\sqrt{2}}{24}. \end{aligned}$$

where we did the substitution $t = 8u^2 + 1$.

2. Show that the surface area of the lateral surface of a circular cone with radius R and height H is $A = \pi R\sqrt{R^2 + H^2}$.

Solution. The equation of a circular cone with radius R and height H (with apex at the origin) is

$$\frac{R^2}{H^2} z^2 = x^2 + y^2, \qquad z \ge 0.$$

We parametrize the cone as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(r,\theta) \in \mathbb{R}^2 \mid r \in [0,R], \theta \in [0,2\pi]\}$$

and

$$\alpha(r,\theta) = \left(r\cos(\theta), r\sin(\theta), \frac{rH}{R}\right).$$

The tangent vectors are

$$\mathbf{T}_r = \left(\cos(\theta), \sin(\theta), \frac{H}{R}\right), \qquad \mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 0)\,.$$

The cross product is

$$\mathbf{T}_r \times \mathbf{T}_\theta = \left(-\frac{Hr}{R}\cos(\theta), -\frac{Hr}{R}\sin(\theta), r\right).$$

Its norm is

$$|\mathbf{T}_r \times \mathbf{T}_\theta| = \sqrt{\frac{H^2r^2}{R^2}\cos^2(\theta) + \frac{H^2r^2}{R^2}\sin^2(\theta) + r^2} = r\sqrt{\frac{H^2}{R^2}+1},$$

since $r \in [0, R]$ and hence is positive. We then calculate the surface area:

$$\begin{aligned}\iint_S \, dS =& \sqrt{\frac{H^2}{R^2}+1}\int_0^R \int_0^{2\pi} r \; d\theta dr \\ =& 2\pi\sqrt{\frac{H^2}{R^2}+1}\int_0^R r \; dr \\ =& \pi R^2\sqrt{\frac{H^2}{R^2}+1} \\ =& \pi R\sqrt{H^2+R^2}.\end{aligned}$$

3. Evaluate the unoriented surface integral

$$\iint_S xy \; dS,$$

where S is the part of the plane $x + 2y + z = 4$ that lies in the first octant.

Solution. We first need to parametrize the surface. We want the part of the plane that lies in the first octant, so we must have $x \geq 0$, $y \geq 0$, and $z \geq 0$. The equation of the plane is $z = 4 - x - 2y$. Since $z \geq 0$ and $y \geq 0$, we know that x cannot be more than 4. So we take $x \in [0, 4]$. Then, since $z \geq 0$, we know that $4 - x - 2y \geq 0$, which means that $2y \leq 4 - x$, that is, $y \leq 2 - \frac{x}{2}$. So we know that $0 \leq y \leq 2 - \frac{x}{2}$.

Now that we identified the region in the (x, y)-plane over which the part of the plane is, we can parametrize it as $\alpha : D \to \mathbb{R}^3$, with

$$D = \{(u, v) \in \mathbb{R}^2 \; | u \in [0, 4], 0 \leq v \leq 2 - \frac{u}{2}\},$$

and

$$\alpha(u, v) = (u, v, 4 - u - 2v)\,.$$

The tangent vectors are

$$\mathbf{T}_u = (1, 0, -1), \qquad \mathbf{T}_v = (0, 1, -2),$$

with cross product

$$\mathbf{T}_u \times \mathbf{T}_v = (1, 2, 1),$$

whose norm is

$$|\mathbf{T}_u \times \mathbf{T}_v| = \sqrt{1 + 2^2 + 1} = \sqrt{6}.$$

We calculate the unoriented surface integral:

$$\begin{aligned}\iint_S xy\, dS =&\sqrt{6} \int_0^4 \int_0^{2-\frac{u}{2}} uv\, dv du\\ =&\frac{\sqrt{6}}{2} \int_0^4 u \left(2 - \frac{u}{2}\right)^2 du\\ =&\frac{\sqrt{6}}{2} \int_0^4 \left(4u - 2u^2 + \frac{u^3}{4}\right) du\\ =&\frac{\sqrt{6}}{2} \left(2(4^2) - \frac{2}{3}4^3 + \frac{4^4}{16}\right)\\ =&\frac{8\sqrt{6}}{3}.\end{aligned}$$

4. Consider the surface $S = \{y = f(x)\} \subset \mathbb{R}^3$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, and $x \in [0, 2]$, $z \in [0, 1]$. Show that the surface area of S is equal to the arc length of the curve $y = f(x)$, $x \in [0, 2]$, in the (x, y)-plane.

Solution. We can parametrize the surface S by $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, v) \in \mathbb{R}^2 \mid u \in [0, 2], v \in [0, 1]\},$$

and

$$\alpha(u, v) = (u, f(u), v).$$

The tangent vectors are

$$\mathbf{T}_u = (1, f'(u), 0), \qquad \mathbf{T}_v = (0, 0, 1),$$

the cross product is

$$\mathbf{T}_u \times \mathbf{T}_v = (f'(u), -1, 0),$$

whose norm is

$$|\mathbf{T}_u \times \mathbf{T}_v| = \sqrt{1 + (f'(u))^2}.$$

The surface area is then

$$\begin{aligned}\iint_S dS &= \int_0^2 \int_0^1 \sqrt{1 + (f'(u))^2}\, dv du\\ &= \int_0^2 \sqrt{1 + (f'(u))^2}\, du.\end{aligned}$$

But this is precisely the formula for the arc length of the parametric curve $\beta : [0, 2] \to \mathbb{R}^2$ with $\beta(u) = (u, f(u))$, which is the curve $\{y = f(x)\} \subset \mathbb{R}^2$ with $x \in [0, 2]$.

By the way, this result is not surprising. Indeed, the surface S is basically just the curve $y = f(x)$ extended uniformly in the z-direction. So the surface area should be the arc length of the curve times the length of the surface in the z-direction. Since S is defined with $z \in [0, 1]$, the length in the z-direction is just 1, so we the surface area should be equal to the arc length of the curve, as we obtained.

5. In single-variable calculus, you saw that the surface area of the solid of revolution obtained by rotating the curve $y = f(x)$, $x \in [a, b]$ (and $a \geq 0$), about the y-axis is given by the definite integral

$$A = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\, dx.$$

Show that this is consistent with our formula for the surface area of parametric surfaces.

Solution. The surface S obtained by rotating the curve $y = f(x)$, $x \in [a, b]$, about the y-axis, can be parametrized as follows. First, we can parametrize the curve $y = f(x)$ in the (x, y)-plane as $(u, f(u), 0)$, $u \in [a, b]$. When we rotate the curve about the y-axis, for a fixed value of u, the point $(u, 0)$ in the (x, z)-plane gets rotated about the y-axis around a circle of radius u. So we should replace $(u, 0)$ by $(u\cos(\theta), u\sin(\theta))$. This gives a parametrization for the surface S as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, \theta) \in \mathbb{R}^2 \mid u \in [a, b], \theta \in [0, 2\pi]\}$$

with

$$\alpha(u, \theta) = (u\cos(\theta), f(u), u\sin(\theta)).$$

The tangent vectors are

$$\mathbf{T}_u = (\cos(\theta), f'(u), \sin(\theta)), \qquad \mathbf{T}_\theta = (-u\sin(\theta), 0, u\cos(\theta)).$$

The cross product is

$$\mathbf{T}_u \times \mathbf{T}_\theta = (uf'(u)\cos(\theta), -u, uf'(u)\sin(\theta)).$$

Its norm is

$$|\mathbf{T}_u \times \mathbf{T}_\theta| = \sqrt{u^2(f'(u))^2\cos^2(\theta) + u^2(f'(u))^2\sin^2(\theta) + u^2} = u\sqrt{1 + (f'(u))^2},$$

since $u \in [a, b]$, $a \geq 0$, and hence u is positive. The surface area is then:

$$\begin{aligned}\iint_S dS &= \int_a^b \int_0^{2\pi} u\sqrt{1 + (f'(u))^2} d\theta du \\ &= 2\pi \int_a^b u\sqrt{1 + (f'(u))^2}\, du,\end{aligned}$$

which is the formula that you obtained in single-variable calculus!

7.3 Applications of unoriented line and surface integrals

While most applications of integration over curves and surfaces involve the oriented line and surface integrals that we studied throughout this course, unoriented line and surface integrals can also be useful in applications, when we want to calculate a quantity associated to a curve or a surface that should not depend on the orientation. In this section we study a few applications of unoriented line and surface integrals, such as calculating the centre of mass of a wire and a thin sheet of material.

Objectives

You should be able to:

- Determine and evaluate unoriented line and surface integrals in the context of applications in science.

7.3.1 Centre of mass of a wire

In the previous sections we studied unoriented line and surface integrals of functions. Those are useful to calculate quantities associated to curves and surfaces that should not depend on a choice of orientation. The prototypical examples were the arc length of a curve and surface area of a surface, but there are many other applications.

Our first application concerns the calculation of the centre of mass of a wire in $\mathbb{R}^n$. Let us first recall the physical concept of centre of mass.

Suppose that there are k point particles of masses $m_1, \dots, m_k$ at positions $X_1, \dots, X_k \in \mathbb{R}$ on a line. The **centre of mass** of the system of particles is located at

$$\bar{x} = \frac{\sum_{i=1}^k m_i X_i}{m},$$

where $m = \sum_{i=1}^k m_i$ is the total mass of the system. The numerator is sometimes called the "first moment of the system about the origin" and written as M.

If we are given instead a rod in $\mathbb{R}$ between $x = a$ and $x = b$ with mass density $\rho(x)$, then to get its centre of mass we use the slicing principle. We slice the rod into small line segments. Let dx be of a typical line segment located at the point x. Its mass is $dm = \rho(x)\ dx$, and its first moment about the origin is $dM = x\rho(x)\ dx$. We then sum over slices and take the limit of an infinite number of infinitesimal slices. This turns the calculation of the total mass and first moment of the rod as a definite integral, and its centre of mass is:

$$\bar{x} = \frac{\int_a^b x\rho(x)\ dx}{\int_a^b \rho(x)\ dx} = \frac{1}{m}\int_a^b x\rho(x)\ dx.$$

Now what if we want to calculate the centre of mass of a wire that is bent and twisted in $\mathbb{R}^n$? We apply the same idea, but thinking of the wire as a parametric curve $\alpha : [a, b] \to \mathbb{R}^n$, with $\alpha(t) = (x_1(t), \dots, x_n(t))$ and image curve $C = \alpha([a, b])$. Let $\rho : C \to \mathbb{R}$ be the mass density of the wire, which we assume to be continuous. We slice the wire (the image curve C) into small curve segments. Let ds (the line element from Subsection 7.1.1) be the length of a typical curve segment located at $\alpha(t)$. Its mass is $dm = \rho(\alpha(t))\ ds$. By summing over all curve segments and taking the limit of an infinite number of segments of infinitesimal length, we calculate the total mass of the wire as an unoriented line integral:

$$m = \int_C \rho\ ds = \int_a^b \rho(\alpha(t))|\mathbf{T}(t)|\ dt.$$

To get the centre of mass, we also need to calculate the first moments of the wire. Here, we are working in $\mathbb{R}^n$. There are n first moments, one in each coordinate $x_1, \dots, x_n$. The first moments of the curve segment are given by

$$x_k(t)\rho(\alpha(t))\ ds, \qquad k = 1, \dots, n.$$

Summing over curve segments, and taking the limit as usual, we obtain that the position of the centre of mass of the wire is given by the point in $\mathbb{R}^n$ with coordinates $\bar{x}_1, \ldots, \bar{x}_n$, with:

$$\bar{x}_k = \frac{1}{m}\int_C x_k \rho \ ds = \frac{1}{m}\int_a^b x_k(t)\rho(\alpha(t))|\mathbf{T}(t)| \ dt, \qquad k = 1, \ldots, n.$$

In other words, to find the centre of mass of the wire, we need to evaluate the unoriented line integrals corresponding to the total mass of the wire and its n first moments.

Note that the centre of mass of the wire will not generally be located on the wire itself, since the wire is twisted and bent in the ambient space $\mathbb{R}^n$; this will be clear in the next example.

Example 7.3.1 Finding the centre of mass of a wire in $\mathbb{R}^2$. A wire is bent into the semi-circle $x^2 + y^2 = 4$, $y \geq 0$. Its mass density is given by the function

$$\rho(x, y) = 4 - y$$

on the wire. Find the centre of mass of the wire.

Let us see what we expect first. The wire is heavier near its base ($y = 0$) than at the top ($y = 2$). The mass density is however symmetric about the y-axis (it does not vary in x). We thus expect the centre of mass to be located at a point $(0, \bar{y})$, with $\bar{y} < 2$, since it should be below the top of the wire.

To calculate the centre of mass, we will need to calulate unoriented line integrals along the curve, so we first parametrize the curve as $\alpha : [0, \pi] \to \mathbb{R}^2$ with

$$\alpha(\theta) = (2\cos(\theta), 2\sin(\theta)).$$

The tangent vector is

$$\mathbf{T}_\theta = (-2\sin(\theta), 2\cos(\theta)),$$

whose norm is

$$|\mathbf{T}_\theta| = \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)} = 2.$$

The line element is then

$$ds = |\mathbf{T}_\theta| d\theta = 2d\theta.$$

We first calculate the total mass of the wire. It is given by the unoriented line integral:

$$\begin{aligned} m =& \int_C \rho \ ds \\ =& \int_0^\pi (4 - 2\sin(\theta))2d\theta \\ =& 4(2\pi - 2) \\ =& 8(\pi - 1). \end{aligned}$$

We then calculate the coordinates $\bar{x}$ and $\bar{y}$ of the centre of mass.

$$\begin{aligned} \bar{x} =& \frac{1}{m}\int_C x\rho \ ds \\ =& \frac{1}{8(\pi - 1)}\int_0^\pi (2\cos(\theta))(4 - 2\sin(\theta))2 \ d\theta \end{aligned}$$

$$=0,$$

since the trigonometric integral vanishes. As for $\bar{y}$, we get:

$$\begin{aligned}
\bar{y} =& \frac{1}{m}\int_C y\rho\ ds \\
=& \frac{1}{8(\pi-1)}\int_0^\pi (2\sin(\theta))(4-2\sin(\theta))2\ d\theta \\
=& \frac{8}{8(\pi-1)}\left(4-\frac{\pi}{2}\right) \\
=& \frac{8-\pi}{2(\pi-1)}
\end{aligned}$$

Therefore, the centre of mass of the wire is located at the point

$$(\bar{x},\bar{y}) = \left(0, \frac{8-\pi}{2(\pi-1)}\right).$$

To make sure that this is consistent with our expectation, we can find a numerical value for the location of the centre of mass. We get:

$$(\bar{x},\bar{y}) \simeq (0, 1.134).$$

This is consistent with our expectation. Phew! □

7.3.2 Centre of mass of a thin sheet

We can do a very similar calculation to obtain the centre of mass of a thin sheet of material (such as aluminium foil, or paper) in $\mathbb{R}^3$. Suppose that the sheet of material takes the shape of a parametric surface $\alpha : D \to \mathbb{R}^3$, with $\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$ and image surface $S = \alpha(D)$. Suppose that $\rho : S \to \mathbb{R}$ is the mass density of the sheet (mass per unit area). Using the same slicing approach, but with small pieces of surface of area dS, we obtain that the total mass of the sheet can be written as the unoriented surface integral

$$m = \iint_S \rho\ dS = \iint_D \rho(\alpha(u,v))|\mathbf{T}_u \times \mathbf{T}_v|\ dA.$$

We calculate the centre of mass as before, by calculating the first moments in the three coordinates x, y, z. Using the slicing process, those become unoriented surface integrals. The result is that position $(\bar{x}, \bar{y}, \bar{z})$ of the centre of mass is given by

$$\begin{aligned}
\bar{x} =& \frac{1}{m}\iint_S x\rho\ dS = \frac{1}{m}\iint_D x(u,v)\rho(\alpha(u,v))|\mathbf{T}_u \times \mathbf{T}_v|\ dA, \\
\bar{y} =& \frac{1}{m}\iint_S y\rho\ dS = \frac{1}{m}\iint_D y(u,v)\rho(\alpha(u,v))|\mathbf{T}_u \times \mathbf{T}_v|\ dA, \\
\bar{z} =& \frac{1}{m}\iint_S z\rho\ dS = \frac{1}{m}\iint_D z(u,v)\rho(\alpha(u,v))|\mathbf{T}_u \times \mathbf{T}_v|\ dA.
\end{aligned}$$

Note that, as was the case for the wire, the centre of mass of the sheet is not generally expected to lie on the sheet itself, since the sheet can be bent and twisted in the ambient space $\mathbb{R}^3$.

Example 7.3.2 Finding the centre of mass of a sheet in $\mathbb{R}^3$. Suppose the a sheet of paper is bent in the shape of the cylinder $x^2 + y^2 = 9$, with $z \in [0, 1]$. Suppose that its mass density is given by the function

$$\rho(x, y, z) = z + 1.$$

Find the centre of mass of the cylinder.

First, let us see what we expect. The cylinder is heavier at the top than at the bottom. However, its mass density has circular symmetry above the z-axis, as it only depends on the height z on the cylinder. Thus we expect the centre of mass to be in the middle of the cylinder, i.e. with coordinates $(0, 0, \bar{z})$. Furthermore, we expect its z-coordinate to be a little bit higher than half-way up the cylinder, since the cylinder is heavier at the top than at the bottom. So we expect $0.5 < \bar{z} < 1$.

To calculate the required unoriented surface integrals, we first parametrize the surface as $\alpha : D \to \mathbb{R}^3$ with

$$D = \{(u, \theta) \in \mathbb{R}^2 \mid u \in [0, 1], \theta \in [0, 2\pi]\}$$

and

$$\alpha(u, \theta) = (3\cos(\theta), 3\sin(\theta), u).$$

The tangent vectors are

$$\mathbf{T}_u = (0, 0, 1), \qquad \mathbf{T}_\theta = (-3\sin(\theta), 3\cos(\theta), 0),$$

and the cross product is

$$\mathbf{T}_u \times \mathbf{T}_\theta = (-3\cos(\theta), -3\sin(\theta), 0).$$

Its norm is

$$|\mathbf{T}_u \times \mathbf{T}_\theta| = \sqrt{9\cos^2(\theta) + 9\sin^2(\theta)} = 3,$$

and thus the surface element is

$$dS = 3du d\theta.$$

We calculate the total mass of the sheet of paper. We get:

$$\begin{aligned} m =& \iint_S \rho \; dS \\ =& \int_0^1 \int_0^{2\pi} (u+1)3 \; d\theta du \\ =& 6\pi \int_0^1 (u+1) \; du \\ =& 9\pi. \end{aligned}$$

As for the coordinates of the centre of mass, we get:

$$\begin{aligned} \bar{x} =& \frac{1}{m} \iint_S x\rho \; dS \\ =& \frac{1}{9\pi} \int_0^1 \int_0^{2\pi} (3\cos(\theta))(u+1)3 \; d\theta du \\ =& 0, \end{aligned}$$

since the integral over θ is zero. Similarly,

$$\begin{aligned}\bar{y} =& \frac{1}{m}\iint_S y\rho \; dS\\ =& \frac{1}{9\pi}\int_0^1\int_0^{2\pi}(3\sin(\theta))(u+1)3 \; d\theta du\\ =& 0.\end{aligned}$$

As for $\bar{z}$, we get:

$$\begin{aligned}\bar{z} =& \frac{1}{m}\iint_S z\rho \; dS\\ =& \frac{1}{9\pi}\int_0^1\int_0^{2\pi}(u)(u+1)3 \; d\theta du\\ =& \frac{2}{3}\int_0^1 (u^2+u) \; du\\ =& \frac{5}{9}.\end{aligned}$$

As a result, the centre of mass of the sheet of paper is located at the point

$$(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{5}{9}\right).$$

Since $\frac{5}{9} \simeq 0.556$,, this is consistent with our expectation that the centre of mass should be on the z-axis, a little bit higher than half-way up the cylinder, which is at $z = 0.5$. $\square$

7.3.3 Exercises

1. Find the centre of mass of a wire that is bent in the shape of a circle of radius R centered at the origin in $\mathbb{R}^2$, with mass density $\rho(x,y) = 2R - y$. Is your result consistent with your expectations?

Solution. Let us first see what we expect. As the wire is bent in a circle centered at the origin, if the mass density was constant (or symmetric about the x- and y-axes), the centre of mass would be at the origin. However, the mass density is not constant here: it decreases linearly with y. It is constant in the x-direction, so we expect that centre of mass to lie on the y-axis. As the mass decreases as y increases, we expect the centre of mass to be at a position $(0,\bar{y})$, with $-R < \bar{y} < 0$.

We parametrize the circle of radius R as $\alpha : [0,2\pi] \to \mathbb{R}^2$ with $\alpha(\theta) = (R\cos(\theta), R\sin(\theta))$. The tangent vector is

$$\mathbf{T}(\theta) = (-R\sin(\theta), R\cos(\theta)).$$

Its norm is

$$|\mathbf{T}(\theta)| = \sqrt{R^2\sin^2(\theta) + R^2\cos^2(\theta)} = R.$$

The total mass of the wire is

$$m = \int_C \rho \; ds = \int_0^{2\pi}(2R - R\sin(\theta))Rd\theta$$

$$=4\pi R^2.$$

The $\bar{x}$-coordinate of the centre of mass is

$$\begin{aligned}\bar{x} =&\frac{1}{m}\int_C x\rho\ ds\\ =&\frac{1}{4\pi R^2}\int_0^{2\pi} R\cos(\theta)(2R - R\sin(\theta))R\ d\theta\\ =&0,\end{aligned}$$

as expected. As for the $\bar{y}$-coordinate, we get:

$$\begin{aligned}\bar{y} =&\frac{1}{m}\int_C y\rho\ ds\\ =&\frac{1}{4\pi R^2}\int_0^{2\pi} R\sin(\theta)(2R - R\sin(\theta))R\ d\theta\\ =&\frac{1}{4\pi R^2}(-R^3\pi)\\ =&-\frac{R}{4}.\end{aligned}$$

Therefore, the position of the centre of mass is

$$(\bar{x}, \bar{y}) = \left(0, -\frac{R}{4}\right).$$

As expected, it is on the y-axis, and the y-coordinate is between $-R$ and 0.

2. Given a wire with density $\rho(x, y)$ that lies on a curve $C \subset \mathbb{R}^2$, its moments of inertia about the x- and the y-axes are defined by

$$I_x = \int_C y^2\rho\ ds, \qquad I_y = \int_C x^2\rho\ ds.$$

Find the moments of inertia of a wire that lies along the line $2x + y = 5$ between $x = 0$ and $x = 1$, with density $\rho(x, y) = x$.

Solution. We parametrize the line as $\alpha : [0, 1] \to \mathbb{R}^2$ with

$$\alpha(t) = (t, 5 - 2t).$$

The tangent vector and its norm are:

$$\mathbf{T}(t) = (1, -2), \qquad |\mathbf{T}(t)| = \sqrt{1 + 2^2} = \sqrt{5}.$$

We can then evaluate the moments of inertia. First,

$$\begin{aligned}I_x =&\int_C y^2\rho\ ds\\ =&\sqrt{5}\int_0^1 (5 - 2t)^2 t\ dt\end{aligned}$$

$$
\begin{aligned}
&=\sqrt{5}\int_0^1 (25t-20t^2+4t^3)\ dt\\
&=\sqrt{5}\left(\frac{25}{2}-\frac{20}{3}+1\right)\\
&=\frac{41\sqrt{5}}{6}.
\end{aligned}
$$

Second,

$$
\begin{aligned}
I_y &= \int_C x^2\rho\ ds\\
&=\sqrt{5}\int_0^1 t^3\ dt\\
&=\frac{\sqrt{5}}{4}.
\end{aligned}
$$

3. Consider a sheet of aluminium foil that is bent in the shape of a paraboloid $z=x^2+y^2$ with $z\in[0,1]$. Suppose that it mass density is $\rho=k$ where k is a constant. Find the total mass of the sheet and its centre of mass. Does it agree with your expectation?

Solution. As the mass density is constant, and the paraboloid has circular symmetry about the z-axis, we expect the centre of mass to lie on the z-axis, somewhere between $z=0$ and $z=1$.

We parametrize the paraboloid as $\alpha : D\to\mathbb{R}^3$ with

$$D=\{(r,\theta)\in\mathbb{R}^2 \mid r\in[0,1],\theta\in[0,2\pi]\}$$

and

$$\alpha(r,\theta)=(r\cos(\theta),r\sin(\theta),r^2).$$

The tangent vectors are

$$\mathbf{T}_r=(\cos(\theta),\sin(\theta),2r),\qquad \mathbf{T}_\theta=(-r\sin(\theta),r\cos(\theta),0).$$

The cross product is

$$\mathbf{T}_r\times\mathbf{T}_\theta=\left(-2r^2\cos(\theta),-2r^2\sin(\theta),r\right).$$

Its norm is

$$|\mathbf{T}_r\times\mathbf{T}_\theta|=\sqrt{4r^4\cos^2(\theta)+4r^4\sin^2(\theta)+r^2}=r\sqrt{4r^2+1}.$$

We first calculate the total mass. We get:

$$
\begin{aligned}
m &= \iint_S \rho\ dS\\
&=k\int_0^1\int_0^{2\pi} r\sqrt{4r^2+1}\ d\theta r\\
&=2\pi k\int_0^1 r\sqrt{4r^2+1}\ dr
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\pi k}{4} \int_1^5 \sqrt{u}\, du \\
&=\frac{\pi k}{6}\left(5^{3/2}-1\right).
\end{aligned}
$$

In the process of evaluating the integral we did the substitution $u = 4r^2 + 1$.

Next we calculate the coordinates of the centre of mass. First,

$$
\begin{aligned}
\bar{x} =&\frac{1}{m} \iint_S x\rho \, dS \\
=&\frac{k}{m} \int_0^1 \int_0^{2\pi} r^2 \cos(\theta)\sqrt{4r^2+1}\, d\theta r \\
=&0,
\end{aligned}
$$

as expected. Next,

$$
\begin{aligned}
\bar{y} =&\frac{1}{m} \iint_S y\rho \, dS \\
=&\frac{k}{m} \int_0^1 \int_0^{2\pi} r^2 \sin(\theta)\sqrt{4r^2+1}\, d\theta r \\
=&0,
\end{aligned}
$$

as expected. Finally,

$$
\begin{aligned}
\bar{z} =&\frac{1}{m} \iint_S z\rho \, dS \\
=&\frac{6k}{\pi k\left(5^{3/2}-1\right)} \int_0^1 \int_0^{2\pi} r^3\sqrt{4r^2+1}\, d\theta r \\
=&\frac{12}{\left(5^{3/2}-1\right)} \int_0^1 r^3\sqrt{4r^2+1}\, dr \\
=&\frac{3}{8\left(5^{3/2}-1\right)} \int_1^5 (u-1)\sqrt{u}\, du \\
=&\frac{3}{8\left(5^{3/2}-1\right)} \left(\frac{2}{5}5^{5/2}-\frac{2}{5}-\frac{2}{3}5^{3/2}+\frac{2}{3}\right) \\
=&\frac{1}{10}\frac{5^{5/2}+1}{5^{3/2}-1}.
\end{aligned}
$$

As before, to evaluate the integral we did the substitution $u = 4r^2 + 1$.

Therefore, the centre of mass is located at

$$
(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{1}{10}\frac{5^{5/2}+1}{5^{3/2}-1}\right) \simeq (0, 0, 0.559).
$$

As expected, it is located on the z-axis, somewhere between $z = 0$ and $z = 1$.

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