

lecture 3

alternating linear clobber

	outcome	class	comment
game A2	ox	N	ox = { 0 0 } called *
game A4	oxox	N	= { up, 0, * down, 0, * }
game A6	oxoxox	P	
game A8	oxoxoxox	?	

outcome class for A6 ? show Left options at root

A6

xxoxo

oxo_xx

oxx_ox

o_xxox

x_oxox

good move for Right?

xo_xo

_oo_xx

_ox_ox

o_xx_o

o_xx_o

so A6 is a P-position

alternating linear clobber conjecture:

A_{2t} (for all positive $2t$, except $2t = 6$) is a N-position

with your neighbour: try to prove that A_8 is an N-position

motivation: how can we reduce our work

in trying to prove that A_8 is an N-position?

first moves from A8 xoxoxoxo

say x plays first, can move to

xoxoxo, xo xxoxo, xoxo xxo, xoxoxo x, xx oxoxo, xoxx oxo, xoxoxx o

definition of games equal to 0 (in cgt)

we say a game equals 0 if and only if it is a P-position

why ?

suppose we have a comb. game g and it decomposes
into 2 independent parts, x and y , and x is a P-psn

we write $g = x + y$

now assume y is an N position.

what can we say about g ?

answer: g is an N-position.

proof?

proof: assume L plays first.

make a winning move on y.

if the opponent replies on y, continue with a winning move on y.

whenever the opponent replies on x, play a 2nd-player wins

strategy on x. this must leave (on x) either 0 or a P-psn (why).

so argue by induction.

recall: in game notation, 0 is the game { | }

now we can simplify: ignore P-positions

outcome class for A6 ? show Left options at root

A6

xxoxo oxo_xx oxx_ox o_xxox x_oxox

simpler: xxoxo oxo oxx + ox xxox oxox

blah blah blah A4 + up + oxoxo + A6 + *

= A4 + up + oxoxo + *

- Devos/Kent chap 1

<https://webdocs.cs.ualberta.ca/~hayward/cgt/asn/devoschap1.pdf>

- rules of go

<https://webdocs.cs.ualberta.ca/~hayward/355/rules.pdf>

lecture 4

- rules of hex

<https://webdocs.cs.ualberta.ca/~hayward/355/hexgame.pdf>

hex

- no draw
- extra stones
- 1pw
- 2pw (irregular)
- virtual connections, mustplay

hex talk

<https://webdocs.cs.ualberta.ca/~hayward/talks/twist2.pdf>

hex page on CMPUT 355

<https://webdocs.cs.ualberta.ca/~hayward/355/jem/hex.html>

hex-the-full-story via ualberta ccid

<https://www.taylorfrancis.com/books/hex-inside-ryan-hayward-bjarne-toft/10.1201/9780429031960>

chomp

Fred Schuh 1952, David Gale 1974, Martin Gardner 1973

<https://en.wikipedia.org/wiki/Chomp>

<https://www.win.tue.nl/~aeb/games/chomp.html>

boxoff

<https://www.abstractgames.org/boxoff.html>

nim

two players. a *pile* is a non-empty set of stones. a *position* is a (possibly empty) set of piles of stones. on a turn, the player-to-move removes a positive number of stones from some pile. whoever cannot make a move loses and the other player wins.

we can represent a position as a (possibly empty) multiset of positive integers. here is a game between A and B starting from $\{3, 5, 5\}$:

$\{3, 5, 5\}$

A $\{3, 1, 5\} = \{1, 3, 5\}$

B $\{1, 3\}$

A $\{1, 1\}$

B $\{1\}$

A $\{\}$

B loses

<https://en.wikipedia.org/wiki/Nim>

<https://webdocs.cs.ualberta.ca/~hayward/355/jem/nim.html>

normal play end condition: whoever cannot move is loser

misère end condition: whoever cannot move is winner

N (next) is the player-to-move next. P (previous) is the player who is not N, i.e. the player-who-moved previously, if there was a previous move.

who wins nim $\{0,0\}$?

N has no legal moves, so loses, so P wins.

who wins nim $\{0,1\} = \{1,0\}$?

N: she can move to the P-position $\{0,0\}$

who wins nim $\{0,n\}$ for any positive n ? N

who wins nim $\{n,n\}$ for any positive n ? P

defn: tweedledee-tweedledum strategy a 2nd-player strategy for player B in which B always mirrors A's move. e.g. in 2-pile nim from $\{n,n\}$, here is such a strategy for second player B: if A just removed t stones from a pile, remove t stones from the other pile

how many moves from nim (1, 3, 3)?

$$1 + 3 + 3 = 7$$

how many *pairwise non-isomorphic* moves from nim (1, 3, 3)?

4: to any of {0,3,3}, {1,2,3}, {1,1,3}, {0,1,3}

how many moves from nim {1, 3, 3}?

4

who wins nim {1, 2, 3}?

solve from bottom up

{0, 0, 0}	P-position	no legal moves
{0, 0, 1}	N-position	exists move to P-position {0,0,0}
{0, 0, 2}	N	exists move to P-position
{0, 1, 1}	P	only move is to N-position
{0, 0, 3}	N	exists move to P-position
{0, 1, 2}	N	exists move to P-position
{1, 1, 1}	N	exists move to P-position
{0, 1, 3}	N	exists move to P-position

$\{0, 2, 2\}$ P

$\{1, 1, 2\}$ N

$\{0, 2, 3\}$ N

$\{1, 1, 3\}$ N

$\{1, 2, 2\}$ N

each move is to N-position

exists move to P-position

exists move to P-position

exists move to P-position

exists move to P-position

$\{1, 2, 3\}$?

nim-sum of multiset of non-neg. integers is xor-sum

e.g. $1 \oplus 3 \oplus 6 \oplus 7 = 3$

1				1
3			1	1
6		1	1	0
7		1	1	1

nim-sum	0	1	1	

multiset $M = \{6, 13, 13, 24, 30\}$ has nim-sum 0:

6			1	1	0
13		1	1	0	1
13		1	1	0	1
24	1	1	0	0	0
30	1	1	1	1	0

nim-sum	0	0	0	0	0

if we change (decrease, increase) one integer in M ?

nim-sum will no longer be 0. do you see why?

6	1	1	0		
10	1	0	1	0	<- changed
13	1	1	0	1	
24	1	1	0	0	0
30	1	1	1	1	0

nim-sum	0	0	1	1	1 <- no longer 0

Bouton's theorem

nimsum of a multiset of positive integers is the columnwise mod-2 sum of the binary representations. Bouton's theorem: a nim position is winning if and only if its nimsum is positive.

e.g. nimsum of $\{3,5,5\}$ is 0b 011 which is 3 which is positive, so Bouton's theorem says that this position is winning (for the player-to-move), i.e. that the ptm has a winning strategy.

3	0	1	1
5	1	0	1
5	1	0	1
nimsum	0	1	1

We can use Bouton's theorem to find all winning moves. $\text{nim}\{3,5,5\}$ is winning, so there must be a move that leaves a losing position, i.e. a position with nimsum 0. here are all positions that ptm can move to from $\text{nim}\{3,5,5\}$: 55 155 255 35 135 235 335 345. compute the nimsum of each. which have nimsum 0?

using Bouton's theorem to find all winning moves

Here is a faster way to use Bouton's theorem to find all winning moves from a nim position:

- in the nimsum array, find the left-most column c whose mod-2 sum is positive (if the nimsum is positive, why must such a column exist?)
- pick any row r with a one in column c . (why must such a row exist?)
- change the 1 at (row r , column c) to 0
- for each location $Z = (r, c')$ with $c' \neq c$, set Z to be the mod-2 sum of all other elements of that column
- why does this process leave a position with nimsum 0?
- why does process correspond to a legal move in nim?

e.g. $\text{nim}\{3,5,5\}$, the only column with mod-2 sum 1 is the middle column, so c is 2 (middle column). the only row with a 1 in that

column is column 1 (top), we set that location to 0. now we set location (row 1, column 3) to the mod-2 sum of the rest of that column which is $\oplus 0^*$. So the only winning move from $\{3,5,5\}$ is to $\{5,5\}$, i.e. from the pile with 3 stones, remove 3 stones.

3		1	1	-->	0	0	0		
5		1	0	1		1	0	1	5
5		1	0	1		1	0	1	5
nimsum		0	1	0		0	0	0	

another example: from $\{3,5,7\}$, three winning moves

3		1	1			1	1	3		1	1	3	-->	1	0	2			
5		1	0	1		1	0	1	5	-->	1	0	0	4		1	0	1	5
7		1	1	1	-->	1	1	0	6		1	1	1	7		1	1	1	7
		0	0	1		0	0	0			0	0	0			0	0	0	

nim-sum theorem (Bouton's theorem) nim state winning iff its
nim-sum is positive

- Let M be a multiset of non-negative integers including a positive integer m . Let M' be the multiset obtained from M by replacing m (just one instance) with an integer $0 \leq m' < m$. If $\text{nim-sum}(M) = 0$, then $\text{nim-sum}(M') \neq 0$.
- Let M be a multiset of non-negative integers such that $\text{nim-sum}(M) \neq 0$. Then some m in M can be replaced with a smaller integer m' so that $\text{nim-sum}(M') = 0$.

How to find m and m' ? Let j be the position of the leftmost 1 in $\text{binary}(\text{nim-sum}(M))$. Let m be any integer in M whose position- j digit in $\text{binary}(m)$ is also 1. Let $M - m$ be the multiset obtained by removing (one instance of m) from M . Let $q = \text{nim-sum}(M - m)$. Let M' be the multiset obtained from M by replacing (one instance of) m with q . Then $\text{nim-sum}(M') = 0$.

example of second part of theorem.

6	1	1	0	
10	1	0	1	0
13	1	1	0	1
24	1	1	0	0
30	1	1	1	1

nim-sum	0	0	1	1
	j			

<- change

j is position 3 (counting from right), so m can be 6, 13, or 30. Pick $m = 13$. Then $M - m = \{6, 10, 24, 30\}$, $\text{nim-sum}(M - m) = 01010_b = 10$,
 $M' = \{6, 10, 10, 24, 30\}$.

6	1	1	0	
10	1	0	1	0
10	1	0	1	0
24	1	1	0	0
30	1	1	1	1

nim-sum 0 0 0 0 0

nim: Bouton's corollary

recall Bouton's theorem:

nimsum-0, all moves to nimsum-pos

nimsum-pos, some move to nimsum-0

corollary nim position winning iff nimsum-pos

proof induction on pile-sum.