

rock paper scissors

play 5 games with your neighbour

minimax value of 5-game rps ? -5

because your maximum guaranteed result

in each game is -1 (loss)

RPS as a matrix game

below is RPS payoff matrix for Rose (row player)

Rose payoff matrix		Colin action		
Rose action		rock	paper	scissors
rock		0	-1	1
paper		1	0	-1
scissors		-1	1	0

matrix game

0-sum game: for each terminal state, $P1\text{-score} + P2\text{-score} = 0$

matrix game 2-player, simultaneous-move, 0-sum

another matrix game

Rose payoff matrix Colin action

Rose action a b c

 a 3 -2 1

 b 0 4 -5

 c -1 -2 -3

Colin payoff matrix Colin action

Rose action a b c

 a -3 2 -1

 b 0 -4 5

 c 1 2 3

by 0-sum def'n,

Colin-score = - Rose-score

stochastic algorithm

stochastic: with action(s) taken from a probability distribution

e.g. stochastic strategy S: play rck,ppr,scr with probabilities .5 .3 .2

- assume Colin plays S: what is Rose's best replying strategy T ?

- first question: best 1-choice T ? analysis below

rck: expected R-winrate: $.5 * 0 + .3 * -1 + .2 * 1 = -.1$

ppr: expected R-winrate: $.5 * 1 + .3 * 0 + .2 * -1 = .3$ <-- best

scr: expected R-winrate: $.5 * -1 + .3 * 1 + .2 * 0 = -.2$

- next question: best stochastic T ?

theorem in a matrix game, every fixed stochastic strategy
has a best counter-strategy that is 1-choice

to find a best stochastic response (counter-strategy),
it suffices to consider only 1-choice responses woo hoo :)

- $m \times n$ matrix game has only
 m 1-choice strategies for R
 n 1-choice strategies for C

continue: best R-response to C-strat (rck .5, ppr .3, scr .2) ?

answer

- assume R plays (rock, paper, scissors) with probability (r, p, s)

$$\text{so } 0 \leq r, p, s \leq 1 \quad r + p + s = 1$$

- by theorem, sufficient to consider 1-choice strategies for R
- already seen: strats $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ R-exp-payoffs $-.1$, $.3$, $-.2$,
an R-best stochastic response is 1-choice strat $(0, 1, 0)$

ttt vs rps

2-player

2-player

alternate turn

simultaneous move

deterministic algorithms

stochastic algorithms

analysis: minimax

analysis: minimax ?

how to find matrix game minimax strategy ?

warmup: 2×2 matrix game

find matrix game value

- our story so far ...
- matrix game value a.k.a. *Von Neumann equilibrium*
- Von Neumann's theorem: every matrix game has a value ...
- today: how to use linear programming to find that value
- you might find it helpful to read this:

Game Theory, A Playful Intro (Kent/Devos), chapter 5.1

matrix game Kent/Devos GT:playful intro, Ch. 5

C		what happens with 1-choice strat	r1 vs c1 ?
2	-1		r2 vs c1 ?
R	1	2	r1 vs c2 ?
			r2 vs c2 ?

now consider R plays mixed strategy $x * r1 + y * r2$,

where x, y are probabilities ($0 \leq x, y \leq 1$ $x + y = 1$)

case 1) $x * r1 + y * r2$ versus c1 ?

case 2) $x * r1 + y * r2$ versus c2 ?

case 1) R exp. payoff $x * 2 + y * 1$

case 2) R exp. payoff $x * -1 + y * 2$

Rose wants minimax stoch. strat

- **minimax** could be called maximin

- for a fixed stoch. R-strat (x, y) , a C-best response?

R-exp-payoff-minimizing (over all 1-choice C-strats)

here $\min\{ 2x + y, -x + 2y \}$

- R wants max (over all R-strats) best C-response (R-exp-payoff minimizing)

- sometimes called R's best guaranteed expected payoff

- here: $\max (\text{over } (x,y)) \min\{ 2x + y, -x + 2y \}$

$$\max \text{ (over } (x,y)) : \min\{ 2x + y, -x + 2y \}$$

maximize z such that

$$z \leq 2x + y$$

$$z \leq -x + 2y$$

$$0 \leq x, y \leq 1$$

$$x + y = 1$$

how to solve 2-dimensional linear program

- try boundary of (x,y) -feasible region
- $(0,0)$: $z = 0$
- $(0,1)$: $z = \min \{2x + y = 1, -x + 2y = 2\} = 1$
- $(1,0)$: $z = \min \{2x + y = 2, -x + 2y = -1\} = -1$
- $(0,1)$ to $(1,0)$ along $x + y = 1$?
- try $x + y = 1$ and $2x + y = -x + 2y$? see next page
- $(1/4, 3/4)$: $z = \min \{2x + y = 5/4, -x + 2y = 5/4\} = 5/4$
- so R's minimax strat is $(1/4, 3/4)$

$$2x + y = -x + 2y$$

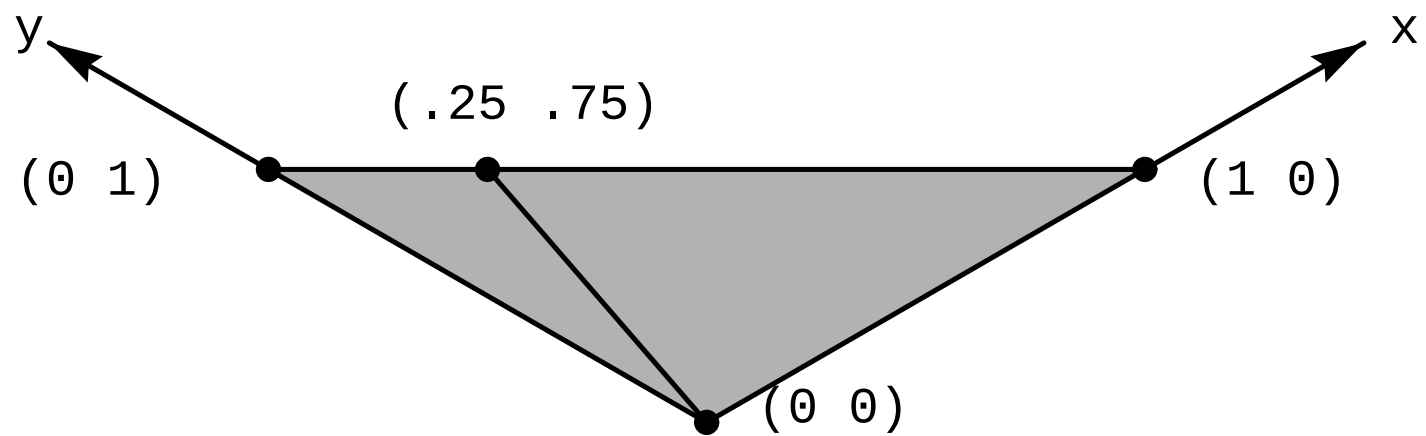
$$3x = y$$

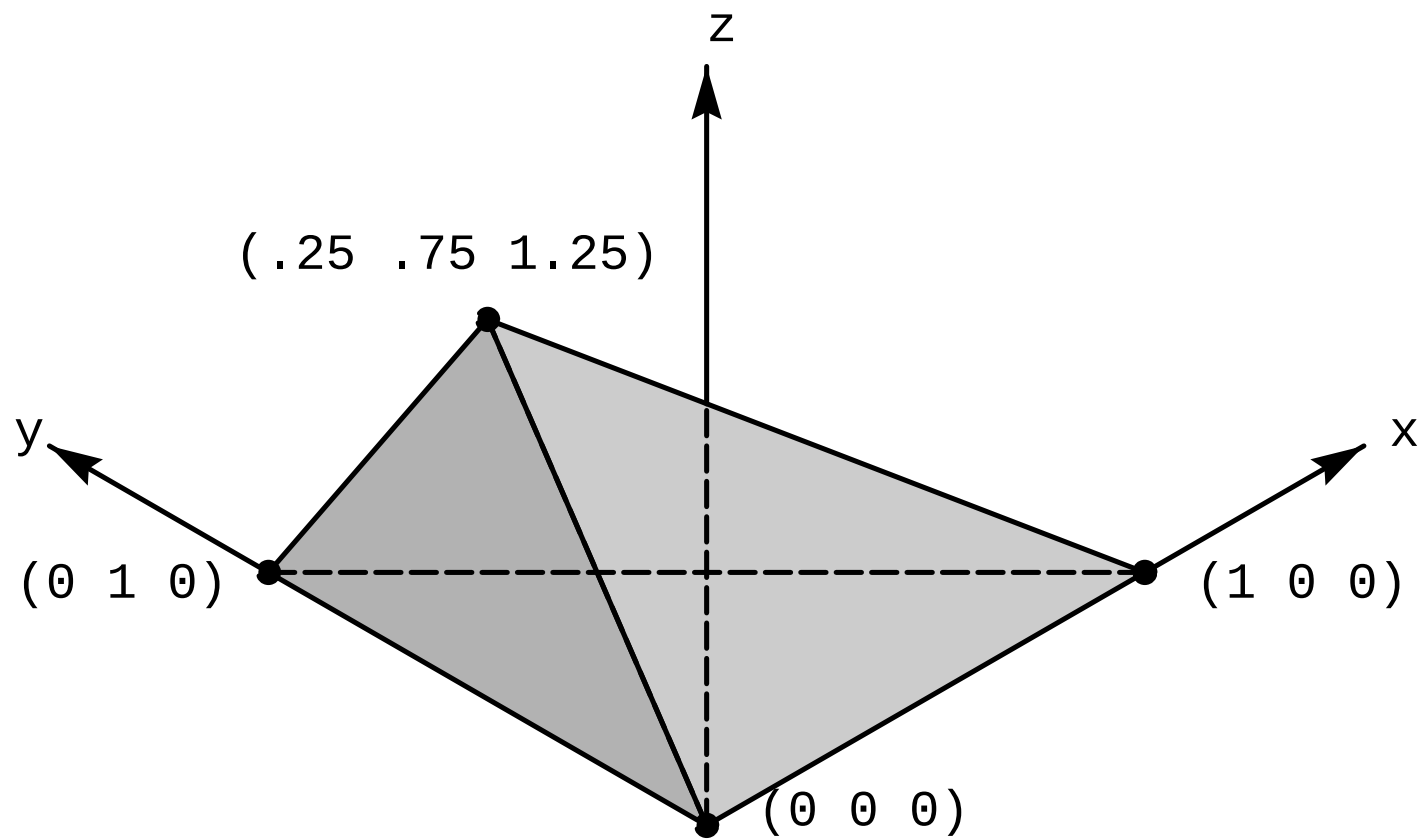
$$x + y = 1$$

$$x + 3x = 1$$

$$4x = 1$$

$$x, y = 1/4, 3/4$$





2x2 matrix game conclusion

- R has a minimax strat, expected payoff $5/4$
- exercise: show C has a minimax strat $(3/4, 1/4)$, exp.-R-payoff $5/4$
- so R can play so that R has exp payoff at least $5/4$
- so C can play so that R has exp payoff at most $5/4$
- so $5/4$ is called the (minimax) value of this game
- this pair of minimax strats called Von Neumann equilibrium

theorem: every matrix game has a VN equilibrium

to solve n-dimensional linear program, use LP-solver

- evaluate at <https://sagecell.sagemath.org/>

```
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
x, y, z = v['x'], v['y'], v['z']
p.set_objective(z)
p.add_constraint(z <= 2*x + y)
p.add_constraint(z <= -x + 2*y)
p.add_constraint(x + y == 1)
p.add_constraint(x >= 0)
p.add_constraint(y >= 0)
p.solve()
p.get_values(z,x,y)
```

you should get this output [1.25, 0.25, 0.75]

- this tells us that R is guaranteed expected payoff 1.25 when she plays row 1 with probability .25 and row 2 with probability .75
- check this: her expected payoff against 1-choice strat column 1? column 2?
- can we verify that this is her maximum guaranteed expected payoff?
- yes: Von Neumann's theorem, which tells us that there will be a mixed strategy for Colin with guaranteed expected payoff (owing to Rose) at most 1.25
- let's use the same method as above to find an optimizing mixed strategy for Colin

- Colin wants to minimize his guaranteed expected payoff
- for any mixed Colin-strat (s,t) we assume Rose will play the maximizing 1-choice R-strat
- Colin wants to minimize $\max\{ 2s - t, s + 2t \}$
- reformulate this as a maximization problem (for SageMath)
- Colin wants to maximize $\min\{ -2s + t, -s - 2t \}$

maximize z such that

$$z \leq -2s + t$$

$$z \leq -s - 2t$$

$$0 \leq s, t \leq 1$$

$$s + t = 1$$

- evaluate this program at <https://sagecell.sagemath.org/>

```
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
s, t, z = v['s'], v['t'], v['z']
p.set_objective(z)
p.add_constraint(z <= -2*s + t)
p.add_constraint(z <= -s - 2*t)
p.add_constraint(s + t == 1)
p.add_constraint(s >= 0)
p.add_constraint(t >= 0)
p.solve()
p.get_values(z,s,t)
```

- you should get this output

`[-1.25, 0.75, 0.25]`

- this tells us that C is guaranteed expected payoff -1.25 when he plays col 1 with prob .75 and col 2 with prob .25
- check this: his expected payoff against 1-choice strat row 1? row 2?
- can we verify that this is his guaranteed expected payoff?
- yes, because R has a guaranteed expected payoff exactly the negative of this amount
- we have found, and verified, that value for this game is 1.25
 - in expected value, by following her mixed $(.25, .75)$ strategy, R is guaranteed to win at least this amount against any 1-choice col strat
 - in expected value, by following his mixed $(.75, .25)$ strategy, C is guaranteed to lose at most this amount against any 1-choice row strat

- another example

		C		
		1	2	1
R	1	1	0	2
	3	3	1	0

- R wants to maximize her guaranteed expected payoff
- for any mixed R-strat (a,b,c) , assume C plays a minimizing 1-choice C-strat
- R wants to maximize $\min\{ a + b + 3c, 2a + c, a + 2b\}$
- R wants to

maximize z such that

$$z \leq a + b + 3c$$

$$z \leq 2a + c$$

$$z \leq a + 2b$$

$$0 \leq a, b, c \leq 1$$

$$a + b + c = 1$$

- use sagemath to find
 - value of this matrix game
 - a minimax strategy for R
 - a minimax strategy for C

- another example

$$\begin{array}{ccccc} & & & & \text{C} \\ & & & & \\ & 0 & 1 & -2 & \\ \text{R} & -1 & 0 & 1 & \\ & 1 & -1 & 0 & \end{array}$$